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THE ESTIMATION OF EIGENVALUES OF TORSIONAL VIBRATIONS OF THE MECHANICAL SYSTEM WITH VARIABLE STIFFNESS MATRIX

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ABSTRACT

In this paper the forming of the stiffness matrix of a mechanical system on the example of torsional vibration of a drive with flexible coupling and multi stage gear box has been described. First the stiffness matrix of the linearized system has been formed taking into account the mean values of stiffnesses of the elastic elements. For such a linearized system the eigenvalues have been computed. Forming of the stiffness matrix has been described as well for the case when the stiffnesses of the elastic elements are variable functions of the turning angle, i.e. of time. The flexible coupling stiffness is then a nonlinear function described by polynomial while the stiffnesses of teeth of the meshed gears are nonlinear and periodic functions that have been described by Fourier series with corresponding number of members. For the system with the stiffness matrix of which the elements are nonlinear functions of time the computation of the eigenvalues has been performed with the flexible coupling stiffness linearized in vicinity of working point and with stiffnesses of teeth of the meshed gears that have been varied with minimal, mean and maximal values. By comparison of eigenvalues computed for the mechanical system with variable stiffness matrix to the eigenvalues computed for the linearized system the influence of the elastic elements stiffness variability on the eigenvalues of torsional vibrations of the mechanical system has been estimated.

Key words: gear teeth stiffness, stiffness matrix, eigenvalues of torsional vibrations.

1. INTRODUCTION

Variable stiffness matrixes appear in mechanical systems in which the stiffnesses of the elastic connections among the inertial members or of the connections of those members to the surroundings are not constant, but in general arbitrary functions of mutual positions of the inertial members and time. An example of such a system is the drive of a working machine comprising flexible coupling and the multi staged gear box. The stiffness of that coupling is a nonlinear function of the turning angle and is described by an exponential function or a polynomial [1]. The stiffness of teeth of the meshed gears is a periodic function of the turning angle and is described by Fourier series [2]. When estimating the eigenvalues the stiffness of flexible coupling has been linearized in the vicinity of working point while the stiffnesses of teeth of the gears have been varied with minimal, mean and maximal values.

2. TORSIONAL MODEL OF THE MECHANICAL SYSTEM

The torsional model of the mechanical system, taken from [3] is shown in Figure 1a. That system consists of electric motor, flexible coupling and pairs of gears with straight teeth, having moments of inertia I_1, I_2, I_{13}, I_{23}, I_{24}, I_{34}, I_{35}, $I_{45}+I_{46}$ and I_{56}, planetary part of the gear box (gears assumed as rigid) and the rotational mass of a working machine with moments of inertia I_{57} and I_{58}. The system has in total 10 degrees of freedom and the generalised vector is:

$$\{q\} = \{\varphi_{11}, \varphi_{12}, \varphi_{13}, \varphi_{23}, \varphi_{24}, \varphi_{34}, \varphi_{35}, \varphi_{45}, \varphi_{56}, \varphi_{58}\}^T \tag{1}$$

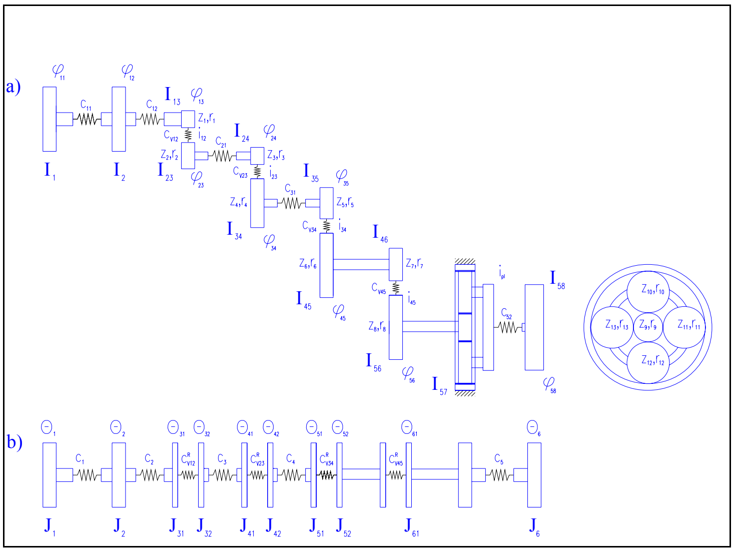
Model in Figure 1a can be reduced to the free torsional chain shown in Figure 1b. The mesh stiffness is represented with an elastic shaft between two discs. The generalized vector of the reduced model is:

$$\{q\} = \{\theta_1, \theta_2, \theta_{31}, \theta_{32}, \theta_{41}, \theta_{42}, \theta_{51}, \theta_{52}, \theta_{61}, \theta_6\}^T, \quad (2)$$

while the relationship between generalized coordinates of both models are the next:

$$\varphi_{11} = \theta_1,\ \varphi_{12} = \theta_2,\ \varphi_{13} = \theta_{31},\ \varphi_{23} = -k_{12}\theta_{32},\ \varphi_{24} = -k_{12}\theta_{41},\ \varphi_{34} = k_{12}k_{23}\theta_{42},\ \varphi_{35} = k_{12}k_{23}\theta_{51},$$
$$\varphi_{45} = -k_{12}k_{23}k_{34}\theta_{52},\ \varphi_{56} = k_{12}k_{23}k_{34}k_{45}\theta_{61},\ \varphi_{58} = -k_{12}k_{23}k_{34}k_{45}k_{pl}\theta_6, \quad (3)$$

k_{ij} is a reduction factor and it is equal to the reciprocal of the gear ratio from the *j-th* to *k-th* shaft.



Moments of inertia of discs on model shown on Figure 1a in [kgm^2] are:

$I_1 = 17$, $I_2 = 29.76$, $I_{13} = 0.68$, $I_{23} = 3.76$, $I_{24} = 0.5$, $I_{34} = 14.76$, $I_{35} = 1.48$, $I_{45}+I_{46} = 93.81$, $I_{56}+I_{57} = 692.8+68.34 = 761.14$, $I_{58} = 212836$.

The stiffnesses of the elastic connections on model shown on Figure 1a are:

$c_{11} = 0.22\times10^6$ Nm/rad, $c_{12} = 4.01\times10^6$ Nm/rad, $c_{v12} = 4.2\times10^9$ N/m, $c_{21} = 38.45\times10^6$ Nm/rad, $c_{v23} = 3.14\times10^9$ N/m, $c_{31} = 53.28\times10^6$ Nm/rad, $c_{v34} = 3.15\times10^9$ N/m, $c_{v45} = 5.20\times10^9$ N/m, $c_{52} = 69.23\times10^6$ Nm/rad

Reduction factors are: $k_{12} = z_1/z_2 = 27/34$, $k_{23} = z_3/z_4 = 23/57$, $k_{34} = z_5/z_6 = 22/78$, $k_{45} = z_7/z_8 = 21/83$, $k_{pl} = r_9/2(r_9+r_{10}) = 0.19512$.

Base radii of the gears [m] are: $r_1 = 0.125$, $r_2 = 0.125$, $r_2 = 0.160$, $r_3 = 0.105$, $r_4 = 0.275$, $r_5 = 0.140$, $r_6 = 0.532$, $r_7 = 0.190$, $r_8 = 0.810$, $r_9 = 0.160$, $r_{10} = 0.250$.

Figure 1. Torsional mechanical model of the drive with gear box

By equalizing the kinetic and potential energy of the model in Figure 1a with the corresponding energies of the model in Figure 1b, the next reduced moments of inertia of the discs are obtained:

$$J_1 = I_1,\ J_2 = I_2,\ J_{31} = I_{13},\ J_{32} = k_{12}^2 I_{23},\ J_{41} = k_{12}^2 I_{24},\ J_{42} = k_{12}^2 k_{23}^2 I_{34},\ J_{51} = k_{12}^2 k_{23}^2 I_{35},$$
$$J_{52} = k_{12}^2 k_{23}^2 k_{34}^2 (I_{45} + I_{46}),\ J_{61} = k_{12}^2 k_{23}^2 k_{34}^2 k_{45}^2 (I_{56} + I_{57}),\ J_6 = k_{12}^2 k_{23}^2 k_{34}^2 k_{45}^2 k_{pl}^2 I_{58}, \quad (4)$$

and the reduced stiffnesses of the elastic connections as well:

$$c_1 = c_{11},\ c_2 = c_{12},\ c_{V12}^R = c_{V12} r_1^2,\ c_3 = k_{12}^2 c_{21},\ c_{V23}^R = k_{12}^2 c_{V23} r_3^2,\ c_4 = k_{12}^2 k_{23}^2 c_{31},$$
$$c_{V34}^R = k_{12}^2 k_{23}^2 c_{V34} r_5^2,\ c_{V45}^R = k_{12}^2 k_{23}^2 k_{34}^2 c_{V45} r_7^2,\ c_5 = k_{12}^2 k_{23}^2 k_{34}^2 k_{45}^2 k_{pl}^2 c_{52}, \quad (5)$$

3. STIFFNESSES OF THE ELASTIC CONNECTIONS

The torsional stiffness of the flexible shafts is calculated using the well known formula from the strength of material. The torsional stiffness of the flexible coupling is calculated according to formula $c_T(\varphi) = dM(\varphi)/d\varphi$. The gear mesh stiffness is a periodic function of a turning angle. The mean value of that stiffness is computed as described in [4] taking into account a unit stiffness, a profile contact ratio and the width of the pinion. In such a manner the gear mesh stiffnesses have been computed in [3]. Graphic representation of the mesh stiffnesses of the gear pairs of the model in Figure 1a are developed in [3] and they are given in Figure 2. These mesh stiffnesses, since they are periodic, can be described by Fourier series:

$$c_v(t) = c_{v0} + \sum_{n=1}^{\infty} (A_n \cos n\Omega t + B_n \sin n\Omega t) \quad (6)$$

here are: $\Omega = \dfrac{\pi n}{30} z$ - gear mesh frequency, $T_\Omega = \dfrac{2\pi}{\Omega}$ - period of the function $c_v(t)$,

c_{v0}, A_n and B_n - constants, n – rotational speed of a pinion , z – number of teeth of the pinion

Constants c_{v0}, A_n i B_n are calculated according to Euler's formulas for coefficients of Fourier series:

$$c_{v0} = \frac{1}{T_\Omega}\int_0^{T_\Omega} c_v(t)\,\mathrm{d}t, \qquad A_n = \frac{2}{T_\Omega}\int_0^{T_\Omega} c_v(t)\cos n\Omega t\,\mathrm{d}t, \qquad B_n = \frac{2}{T_\Omega}\int_0^{T_\Omega} c_v(t)\sin n\Omega t\,\mathrm{d}t.$$

Constant c_{v0} is the mean value of $c_v(t)$ and is equal to the stiffness mean value calculated as in [4].

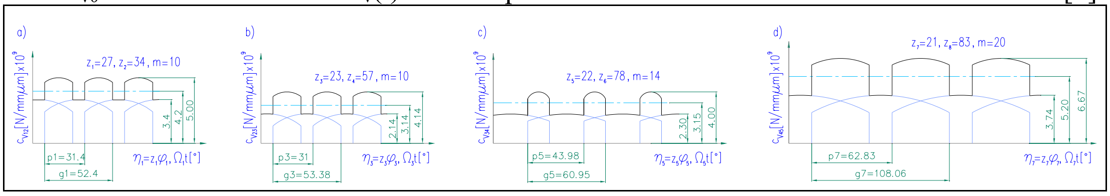


Figure 2. Graphic presentation of gear mesh stiffnesses

Variable mesh stiffness represents the parametric excitation [1,2,5] because of which instability in the working of the system can arise if the frequency of a certain harmonic of the excitation is close to some of the natural frequencies or to some combination of those frequencies [6]. In such a case the most interesting are three types of instability: primary instability $\Omega\approx 2\omega_p$, secondary instability $\Omega\approx\omega_p$ and combined instability with $\Omega\approx\omega_p+\omega_q$, as in [6,7,8].

4. FORMING OF INERTIA AND STIFFNESS MATRICES

By calculating the partial derivations of kinetic and potential energy after generalized velocities and coordinates and by inserting them into the Lagrange's equations of 2[nd] kind the system of differential equations of torsional vibrations of the mechanical system has been obtained, with the matrix of inertia

$$[M] = dijag[J_1, J_2, J_{31}, J_{32}, J_{41}, J_{42}, J_{51}, J_{52}, J_{61}, J_6] \tag{7}$$

and the stiffness matrix:

$$[C]=\begin{bmatrix} c_1 & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & c_1+c_2 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_2 & c_2+c_{V12}^R & -c_{V12}^R & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_{V12}^R & c_3+c_{V12}^R & -c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_3 & c_3+c_{V23}^R & -c_{V23}^R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_{V23}^R & c_4+c_{V23}^R & -c_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_4 & c_4+c_{V34}^R & -c_{V34}^R & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_{V34}^R & c_{V34}^R+c_{V34}^R & -c_{V45}^R & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{V45}^R & c_5+c_{V45}^R & -c_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_5 & c_5 \end{bmatrix} \tag{8}$$

In this matrix the c_{Vij} are periodically variable gear mesh stiffnesses given by equation (6).

5. EIGENVALUES OF THE MECHANICAL SYSTEM

Using the Lagrange's equations of the 2[nd] kind the differential equations of motion are obtained as:

$$[M]\{\ddot{q}\}+[D_V(t)]\{\dot{q}\}+[C_V(t)]\{q\}=\{Q\} \tag{9}$$

here are: $[M]$ matrix of inertia, $[D_V(t)]$ and $[C_V(t)]$ variable damping and stiffness matrix and Q - vector of excitation forces. For free vibration the system of equations (9) is reduced to the system:

$$[M]\{\ddot{q}\}+[C_0]\{q\}=\{0\} \tag{10}$$

By introducing a new matrix $[R]$ such that $[M]=[R]^T[R]$ and the new generalized coordinates $\{\phi\}=[R]\{q\}$, as has been done in [3], the system of equations is obtained in the form of:

$$[I]\{\ddot{\phi}\}+[A]\{\phi\}=\{0\}, \tag{11}$$

where $[I]=[R][R]^{-1}$ and $[A]=[R]^{T^{-1}}[C_0][R]^{-1}$, see [3]. Members of matrix [R] are $R_i=\sqrt{J_i}$.

Assuming solutions in form $\{\phi\}=\{\tilde{q}\}\sin\omega t$ we come to the eigenvalues problem formulated as:

$$(-\omega^2[I]+[A])\{\tilde{q}\}=\{0\} \tag{12}$$

Using subroutines from program set Eispack the eigenvalues of the mechanical system are computed.

6. RESULTS OF CALCULATION

In the table are given the eigenvectors for the first six eigenfrequencies ($\omega^2>0$), computed with mesh stiffnesses mean values, and in Fig. 3. the modes of vibrations for those six eigenfrequencies. The mesh stiffnesses have been varied with minimal c_{vmin}, mean, c_{vsr} and maximal stiffness c_{max} .

	1	2	3	4	5	6	7
ω^2	.788538E-01	.356195E+03	.189720E+05	.581583E+06	.231791E+07	.606805E+07	.964405E+07
1	.545476E+00	.156283E+00	-.823398E+00	.690315E-02	.462874E-03	.582984E-04	-.151523E-04
2	.721723E+00	.201068E+00	.512843E+00	-.402656E+00	-.109434E+00	-.362075E-01	.149683E-01
3	.109096E+00	.302659E-01	.798532E-01	.198613E+00	.267300E+00	.240868E+00	-.159664E+00
4	.203671E+00	.564884E-01	.149314E+00	.398134E+00	.519393E+00	.449484E+00	-.286749E+00
5	.742522E-01	.205533E-01	.547881E-01	.195038E+00	.128632E+00	-.955014E-01	.187696E+00
6	.162840E+00	.450372E-01	.120444E+00	.469778E+00	.217112E+00	-.428362E+00	.621663E+00
7	.515794E-01	.142166E-01	.383167E-01	.178081E+00	-.579716E-01	-.184503E+00	-.645637E-01
8	.115790E+00	.318197E-01	.862982E-01	.450716E+00	-.367584E+00	-.448153E+00	-.616199E+00
9	.834635E-01	.226490E-01	.624565E-01	.382457E+00	-.666816E+00	.557798E+00	.294099E+00
10	.270415E+00	-.962737E+00	-.359357E-02	-.705852E-03	.308682E-03	-.985576E-04	-.327286E-04

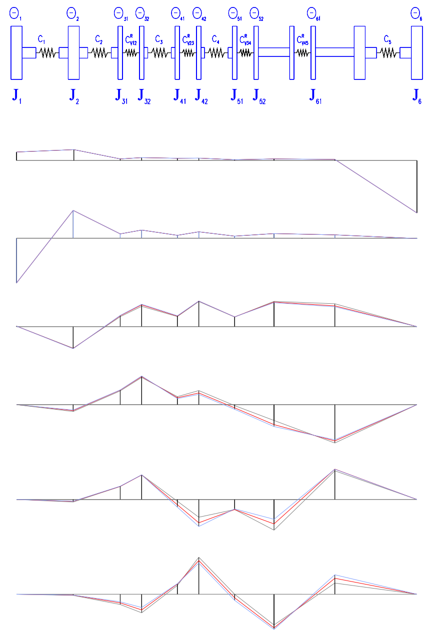


Fig. 3. First six principal modes of vibrations

7. CONCLUSIONS

- The variable teeth mesh stiffness has no influence on first two principal modes of vibrations. By these two modes the rotors of motor and working machine and coupling have the biggest angular displacements.
- The variable teeth mesh stiffness has no significant influence on the other modes either but it has an influence on the magnitudes of eigenfrequencies of each single mode. Those frequencies are considerably lower when computed with c_{vmin} instead of c_{vsr} and slightly higher when computed with c_{max} instead of c_{vsr}.
- Frequency ω_6 is close to the frequency of second harmonic of mesh stiffness of gear pair z_5/z_6, while frequency ω_7 is close to the mesh frequency of gear pair z_3/z_4, etc.
- The other frequencies are substantially higher and they can be placed in the vicinity of some harmonic of mesh stiffness or gear mesh frequency but because of small amplitudes more significant resonances are not possible.

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