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THE STUDY OF THE SPRING WITH A CONCENTRATED VERTICAL LOAD IN ITS PLANE BY TRANSFER-MATRIX METHOD

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ABSTRACT
This work presents a study of the analytical calculus for springs, challanged with a concentrated vertical load in its plane, using the Transfer-Matrix Method. We write the basic equations of the spring theory and the Transfer-Matrix for one spring, challanged in its plane. We are deducted the general expression for the Transfer-Matrix of a round spring, challanged in its plane, with an application for a round spring with a concentrated vertical load in its plane, using the Dirac's and Heaviside's distribution fonctions and operators.
Keywords: state vector, Transfer-Matrix Method, round spring, Dirac's fonction, Heaviside's fonction.

1. INTRODUCTION
The study of the springs is very important for a lot of industrie domains. Using the Transfer-Matrix Method, we can write the basic equations of the spring theory with Dirac's and Heaviside's fonctions and operators and we calculate the six elements of the origin state vector. After, we can calculate, in all spring sections, the state vectors.

2. THE BASIC EQUATIONS OF THE SPRING THEORY
We have studed a circular spring with a constant inertia and have keeped the sign conventions - as well as to the beams - for the internal efforts, for the deplacements and for the exterior loads. The deplacements owing to the cutter force is negleted face to the deplacements due by the flexion moment [1]. We have cosidered a part of a circular spring (Γ), with a mobile reference system (t, n), connected at a current point M (Figure 1, a., [1]).

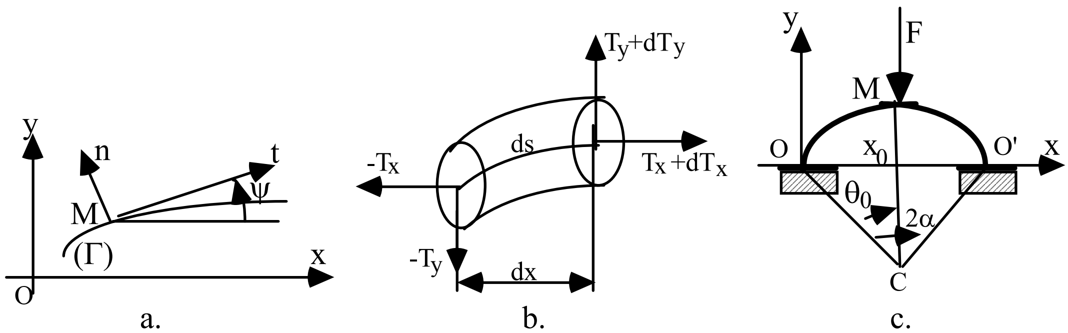

Figure 1.

We have isolated a linie element dx (Figure 1., b.) - a curved element ds - challenged by the exterior loads p(x)dx - after Ox axle and q(x)dx - after Oy axle.
We can write for the element, the balance equations and after integration we have:

$$T_x(\theta)=T_{x0}-p_1(\theta) \quad (1)$$

$T_y(\theta)=T_{y0}-q_1(\theta)$ (2)

$M(\theta)=M_0+T_{x0}g(\theta)-T_{y0}f(\theta)+r(\theta)$ (3)

with: $r(\theta)=\int_0^x [q_1(\theta)f'(\theta)-p_1(\theta)g'(\theta)]d\theta$ (4)

For a constant inertia, after integration, the expressions for ω, u and v are:

$$\omega=\omega_0+\frac{1}{EI}M_0\int hd\theta+\frac{1}{EI}T_{x0}\int ghd\theta-\frac{1}{EI}T_{y0}\int fhd\theta+\frac{1}{EI}\int hrd\theta \quad (5)$$

$$u=u_0-\omega_0 g(\theta)-\frac{M_0}{EI}\int g'\int h-\frac{T_{x0}}{EI}\int g'\int gh+\frac{T_{y0}}{EI}\int g'\int fh-\frac{1}{EI}\int g'\int rh \quad (6)$$

$$v=v_0+\omega_0 f(\theta)+\frac{M_0}{EI}\int f'\int h+\frac{T_{x0}}{EI}\int f'\int gh-\frac{T_{y0}}{EI}\int f'\int fh+\frac{1}{EI}\int f'\int rh \quad (7)$$

3. THE TRANSFER-MATRIX FOR A SPRING, CHALLANGED IN ITS PLANE

For the spring element dx, we can write the expressions (8):

$$\begin{cases} M(\theta)=M_0+T_{x0}g(\theta)-T_{y0}f(\theta)+r(\theta) \\ T_{x0}(\theta)=T_{x0}-p_1(\theta) \\ T_{y0}(\theta)=T_{y0}-q_1(\theta) \\ \omega(\theta)=M_0\frac{1}{EI}\int h+T_{x0}\frac{1}{EI}\int gh-T_{y0}\frac{1}{EI}\int fh+\omega_0+\frac{1}{EI}\int hr \\ u(\theta)=-M_0\frac{1}{EI}\int g'\int h-T_{x0}\frac{1}{EI}\int g'\int gh+T_{y0}\frac{1}{EI}\int g'\int fh-\omega_0 g(\theta)+u_0-\frac{1}{EI}\int g'\int rh \\ v(\theta)=M_0\frac{1}{EI}\int f'\int h+T_{x0}\frac{1}{EI}\int f'\int gh+T_{y0}\frac{1}{EI}\int f'\int fh+\omega_0 f(\theta)+v_0+\frac{1}{EI}\int f'\int rh \end{cases} \quad (8)$$

We have for the spring a state vector at the current point M, $\{U\}_x$, with six elemernts:

$$\{U\}_x=\{M(\theta),T_{x0}(\theta),T_{y0}(\theta),\omega(\theta),u(\theta),v(\theta)\}^{-1} \quad (9)$$

We can write a matrix relation between this state vector, for the section x and the state vector for the section 0, $\{U\}_0$: $\{U\}_x=[T]_x\{U\}_0+\{U_e\}_x$ (10)

where: $[T]_x$ is the transfer-matrix for the passage between the section 0 at the section x and $\{U_e\}_x$ is the state vector for the free term at the section x, or, we can write:

$$\begin{Bmatrix} M(\theta) \\ T_x(\theta) \\ T_y(\theta) \\ \omega(\theta) \\ u(\theta) \\ v(\theta) \end{Bmatrix}=\begin{bmatrix} 1 & g & -f & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{EI}\int h & \frac{1}{EI}\int gh & -\frac{1}{EI}\int fh & 1 & 0 & 0 \\ -\frac{1}{EI}\int g'\int h & -\frac{1}{EI}\int g'\int gh & \frac{1}{EI}\int g'\int fh & -g & 1 & 0 \\ \frac{1}{EI}\int f'\int h & \frac{1}{EI}\int f'\int gh & -\frac{1}{EI}\int f'\int fh & f & 0 & 1 \end{bmatrix}\begin{Bmatrix} M_0 \\ T_{x0} \\ T_{y0} \\ \omega_0 \\ u_0 \\ v_0 \end{Bmatrix}+\begin{Bmatrix} r \\ -p_1 \\ -q_1 \\ \frac{!}{EI}\int rh \\ -\frac{1}{EI}\int g'\int rh \\ \frac{1}{EI}\int f'\int rh \end{Bmatrix} \quad (11)$$

The total transfer-matrix for all sections of the spring is obtained if we give at the parameter θthe correspondent value for the spring end.

4. THE TRANSFER-MATRIX FOR A CIRCULAR SPRING CHALLANGED IN ITS PLANE

We have a circular spring (Figure 1., c.), with a radius R, an angle 2 and the origin of the axels system is O, same with the spring origin.

We write the parameter equations:

$$\begin{cases} x=f(\theta)=R[\sin\alpha+\sin(\theta-\alpha)] \\ y=g(\theta)=R[-\cos\alpha+\cos(\theta-\alpha)] \end{cases} \quad (12)$$

After the calculus and with θ=2α, we have the general transfer-matrix for the spring (13):

$$[T]_x = \begin{bmatrix} 1 & 0 & -2R\sin\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{2R}{EI}\alpha & \frac{2R^2}{EI}(\sin\alpha - \alpha\cos\alpha) & -\frac{2R^2}{EI}\alpha\sin\alpha & 1 & 0 & 0 \\ \frac{2R^2}{EI}(\sin\alpha - \alpha\cos\alpha) & -\frac{R^3}{EI}\left(\frac{3}{2}\sin 2\alpha - \alpha\cos 2\alpha - 2\alpha\right) & \frac{R^3}{EI}(-1 + \cos 2\alpha + \alpha\sin 2\alpha) & 0 & 1 & 0 \\ \frac{2R^2}{EI}\alpha\sin\alpha & \frac{R^3}{EI}(1 - \cos 2\alpha - \alpha\sin 2\alpha) & -\frac{R^3}{EI}\left(\frac{1}{2}\sin 2\alpha - \alpha\cos\alpha\right) & 2R\sin\alpha & 0 & 1 \end{bmatrix} \quad (13)$$

The vector $\{U_e\}_x$ of the relation (11) will be calculated fonction of the exterior density loads.

5. THE CIRCULAR SPRING WITH A CONCENTRATED VERTICAL LOAD CHALLANGED IN ITS PLANE

We study an exemple: a circular spring with a concentrated vertical load F, challanged in its plane (Figure 1., c.).

We can write for the densities, with Dirac's and Heaviside's fonctions and operators:

$$\begin{cases} p(x) = 0 \\ q(x) = -F\delta(x - x_0) \end{cases} \quad (14)$$

We change x by θ and have the expressions (15):

$$\begin{cases} p_1(\theta) = 0 \\ q_1(\theta) = q(x) = -FY(\theta - \theta_0) \\ r(\theta) = -FRY(\theta - \theta_0)[\sin(\theta - \alpha) - \sin(\theta_0 - \alpha)] \end{cases} \quad (15)$$

For θ=2α, we have the general matrix expression for the end O':

$$\begin{Bmatrix} M(2\alpha) \\ T_x(2\alpha) \\ T_y(2\alpha) \\ \varphi(2\alpha) \\ u(2\alpha) \\ v(2\alpha) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -2R\sin\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{2R}{EI}\alpha & \frac{2R^2}{EI}(\sin\alpha - \alpha\cos\alpha) & -\frac{2R^2}{EI}\alpha\sin\alpha & 1 & 0 & 0 \\ \frac{2R^2}{EI}(\sin\alpha - \alpha\cos\alpha) & -\frac{R^3}{EI}\left(\frac{3}{2}\sin 2\alpha - \alpha\cos 2\alpha - 2\alpha\right) & \frac{R^3}{EI}(-1 + \cos 2\alpha + \alpha\sin 2\alpha) & 0 & 1 & 0 \\ \frac{2R^2}{EI}\alpha\sin\alpha & \frac{R^3}{EI}(1 - \cos 2\alpha - \alpha\sin 2\alpha) & -\frac{R^3}{EI}\left(\frac{1}{2}\sin 2\alpha - \alpha\cos\alpha\right) & 2R\sin\alpha & 0 & 1 \end{bmatrix} \begin{Bmatrix} M_0 \\ T_{x0} \\ T_{y0} \\ \varphi_0 \\ u_0 \\ v_0 \end{Bmatrix} +$$

$$+\begin{Bmatrix} -FR[\sin\alpha - \sin(\theta_0 - \alpha)] \\ 0 \\ F \\ -\frac{FR^2}{EI}[\cos(\theta - \alpha) - \cos\alpha - (2\alpha - \theta_0)\sin(\theta_0 - \alpha)] \\ -\frac{FR^3}{EI}\left[1 - \cos(2\alpha - \theta_0) + (2\alpha - \theta_0)\cos\alpha\,\sin(\theta_0 - \alpha) + \frac{\cos 2\alpha - \cos 2(\theta_0 - \alpha)}{4}\right] \\ -\frac{FR^3}{EI}\left[\sin(2\alpha - \theta_0) - \frac{2\alpha - \theta}{2} - (2\alpha - \theta_0)\sin\alpha\sin(\theta_0 - \alpha) - \frac{\sin 2\alpha - \sin 2(\theta_0 - \alpha)}{4}\right] \end{Bmatrix} \quad (16)$$

In this moment, we pose the edge conditions: we have the left edge and the right edge embebded (Figure 1., c.).

The conditions at left edge, in the origin, for θ=0, are:

$$\begin{cases} \omega_0 = 0 \\ u_0 = 0 \\ v_0 = 0 \end{cases} \tag{17}$$

and at right edge, for θ=2α, are:

$$\begin{cases} \omega(2\alpha) = 0 \\ u(2\alpha) = 0 \\ v(2\alpha) = 0 \end{cases} \tag{18}$$

With this conditions and for $\theta_0=45^0$ and $2\alpha=90^0$, we have for the expression (16):

$$\begin{Bmatrix} M(2\alpha) \\ T_x(2\alpha) \\ T_y(2\alpha) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -R\sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\pi R}{2EI} & \frac{R^2\sqrt{2}}{EI}\left(1-\frac{\pi}{4}\right) & -\frac{\pi R^2\sqrt{2}}{4EI} & 1 & 0 & 0 \\ \frac{R^2\sqrt{2}}{EI}\left(1-\frac{\pi}{4}\right) & -\frac{3R^3}{2EI}(3-\pi) & -\frac{R^3}{EI}\left(1-\frac{\pi}{4}\right) & 0 & 1 & 0 \\ \frac{\pi R^2\sqrt{2}}{4EI} & \frac{R^3}{EI}\left(1-\frac{\pi}{4}\right) & -\frac{R^3}{2EI}\left(1-\frac{\pi\sqrt{2}}{4}\right) & R\sqrt{2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} M_0 \\ T_{x0} \\ T_{y0} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -\frac{FR\sqrt{2}}{2} \\ 0 \\ F \\ -\frac{FR^2}{EI}\left(1-\frac{\sqrt{2}}{2}\right) \\ \frac{FR^3}{4EI} \\ -\frac{FR^3}{4EI}\left(2\sqrt{2}-1-\pi\right) \end{Bmatrix} \tag{19}$$

We can write a system with 6 equations and 6 variables. After resolution, the results give the state vectors for the origin face O and for the end O' of the spring.

6. CONCLUSIONS

The Transfer-Matrix Method is very easy to applied for the spring calculus. We can to programm this calculus and with this code, based by the Matrix-Transfer Method, we can calculated rapidly, in the origin section and in the end section of the spring, the 12 elements of the two state vectors. With this results, we can calculated all the state vectors in all sections for the complete spring.

7. REFERENCES

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