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DISCRETIZATION of NON-LINEAR REACTION-DIFFUZION PROBLEM

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ABSTRACT
We consider the singularly perturbed selfadjoint one-dimenzional non-linear reaction-diffusion problem,

$$L_\varepsilon y := \varepsilon^2 y''(x) - py(x) = f(x,y), \quad on \ (0,1)$$

$$y(0)=0\,; \quad y(1)=0\,,$$

where f(x,y) is non-linear function. For this problem, using spline-metod with the natural choice of function, a difference scheme, on a non-uniform mesh, is given. Constructed non-linear difference scheme has uniform convergence in points of the uneven division segments..

Key words. Non-linear reaction-diffusion problem, difference scheme, singular perturbation problem

1. INTRODUCTION

We consider the following non-linear reaction-diffusion problem

$$\varepsilon^2 y''(x) - p\,y(x) = f(x,y), \text{ on } (0,1) \tag{1}$$

$$y(0)=0 \ ; \quad y(1)=0\,, \tag{2}$$

where $p = const > 0$, $0 < \varepsilon < 1$. In general case for non-linear function $f(x,y)$, we suppose that it is coninously differentiable, and has strictly positive derivative by varieable y, that is

$$\frac{\partial f}{\partial y} = f_y \geq m > 0 \quad \text{on} \quad [\ 0,1\ ]\, x\, R \quad \text{(m=const.)} \tag{3}$$

It's clear, from theory of boundary problems, that the reaction-diffusion problem (1)-(3) has unique continuous differentiable solution. The solution y has, in general, a boundary layer at both end points of $[0,1]$.

2. DISCRETIZATION OF THE PROBLEM

Let's write difference equation (1) as

$$L_\varepsilon\, y(x) := \varepsilon^2 y''(x) - p\ y(x) = f(x,y), \quad \text{on } [0,1]. \tag{4}$$

Next, let $0 = x_0 < x_1 < x_2 < ... < x_N = 1$, be a mesh on the interval [0,1].

Consider following boundary problems:

$$L_\varepsilon u_i(x) := 0\ , \ \text{na} \ (x_i, x_{i+1})\ , \ u_i(x_i) = 1\ , u_i(x_{i+1}) = 0\,, \quad (i = 0,1,...,\ N-1) \tag{5a}$$

$$L_\varepsilon u_i(x) := 0\ , \ \text{na} \ (x_i, x_{i+1})\ , \ u_i(x_i) = 0\ , u_i(x_{i+1}) = 1\,, \quad (i = 0,1,...,\ N-1) \tag{5b}$$

It's obvious that problems (5) can be analytically solved. Solutions of the problems (5a) and (5b) denote by $u_i^I(x)$, $u_i^{II}(x)$ $(i = 0,1,...,\ N-1)$, respectively. Note that $u_i^I(x)$ i $u_i^{II}(x)$ are two linearly independent solutions of equation $L_\varepsilon u = 0$, on (x_i, x_{i+1}) $(i = 0,1,...,\ N-1)$.

From previous works, we know functions $u_i^I(x)$ and $u_i^{II}(x)$ (see $[2]$), and they are of the form

$$u_i^I(x)=\frac{\sinh(\beta(x_{i+1}-x))}{\sinh(\beta h_i)} \qquad (x\in[x_i,x_{i+1}])$$

$$(i=0,1,\ldots,\ N-1)\ .$$

$$u_i^{II}(x)=\frac{\sinh(\beta(x-x_i))}{\sinh(\beta h_i)} \qquad (x\in[x_i,x_{i+1}]),$$

where $\beta=\frac{\sqrt{\gamma}}{\varepsilon}$, $h_i=x_{i+1}-x_i$.

Now consider new boundary problem

$$L_\varepsilon\ y_i(x)=\psi\ (x,y_i)\ ,\ \text{ on }\ (x_i,x_{i+1}) \qquad (i=0,1,\ldots,\ N-1) \tag{6}$$

$$y_i(x_i)=y\ (x_i) \qquad ; \qquad y_i(x_{i+1})=y\ (x_{i+1}).$$

It's clear that we have $y_i(x)\equiv y(x)$ on $[0,1]$ $(i=0,1,\ldots,\ N-1)$. Now, we can write solution $y_i(x)$ of the problem 6) in the form

$$y_i(x)=C_1u_i^I(x)+C_2u_i^{II}(x)+\int_{x_i}^{x_{i+1}}G_i(x,s)\ f\ (s,y(s))ds, \quad (x\in[x_i,x_{i+1}]),$$

where $G_i(x,s)$ is Green function for operator L_ε on the interval $[x_i,x_{i+1}]$.

From boundary conditions (6) we have

$$C_1=y(x_i):=y_i\ ,\ \ C_2=y(x_{i+1})=y_{i+1} \quad (i=0,1,\ldots,\ N-1).$$

Thus, solution y_i of the problem (6) on the interval $[x_i,x_{i+1}]$ will have the form

$$y_i(x)=y_iu_i^I(x)+y_{i+1}u_i^{II}(x)+\int_{x_i}^{x_{i+1}}G\ (x,s)\ f\ (s,y(s))ds\ . \tag{7}$$

As it is $y_i(x)\equiv y(x)$ on $[x_i,x_{i+1}]$ ($i=0,1,\ldots,\ N-1$) an *y(x)* is solution which is continously differentiable, that is

$$y_i'(x)|_{x=x_i}=y_{i-1}'(x)|_{x=x_i} \quad \text{for}\ \ (i=1,2,\ldots,\ N-1)\ .$$

Now, by derivating equality (7), considering the last equality, we have

$$y_{i-1}\left(u_{i-1}^I(x)\right)'_{x=x_i}+y_i\left[\ \left(u_{i-1}^{II}(x)\right)'_{x=x_i}-\left(u_i^I(x)\right)'_{x=x_i}\ \right]+y_{i+1}\left[-\left(u_i^{II}(x)\right)'_{x=x_i}\ \right]=$$
$$=\frac{d}{dx}\left[\ \int_{x_i}^{x_{i+1}}G_i(x,s)\ f\ (s,y(s))ds-\int_{x_{i-1}}^{x_i}G_{i-1}(x,s)\ f\ (s,y(s))ds\ \right]_{x=x_i}\ . \tag{8}$$

where $y_k=y(x_k)$ $(k=i-1,i,i+1)$.

Let $\quad a_i=-\left(u_{i-1}^I(x)\right)'_{x=x_i}\ ;\ c_i=\left(u_{i-1}^{II}(x)\right)'_{x=x_i}-\left(u_i^I(x)\right)_{x=x_i}\ ;\ b_i=-\left(u_i^I(x)\right)'_{x=x_i}$.

Now, if we calculate a_i , b_i , c_i $(i=0,1,\ldots,\ N-1)$, we have

$$a_i=\frac{\beta}{\sinh(\beta h_{i-1})}\ ;\ b_i=\frac{\beta}{\sinh(\beta h_i)}\ ;\ c_i=\frac{\beta}{\tanh(\beta h_{i-1})}+\frac{\beta}{\tanh(\beta h_i)}.$$

Thus, after differentiation of right side in last equality and some assortment, (8) takes the form

$$a_iy_{i-1}-c_iy_i+b_iy_{i+1}^I=\frac{1}{\varepsilon^2}\left[\int_{x_{i-1}}^{x_i}u_{x_{i-1}}^{II}(s)\ f(s,y(s))ds+\int_{x_i}^{x_{i+1}}u_{x_i}^I(s)\ f(s,y(s))ds\right],$$
$$y_0=0\quad;\quad y_N=0\quad \text{for}\ \ (i=1,\ldots,\ N-1)\ . \tag{9}$$

Difference scheme (9) gives exact solution of the problem, in points of a fixed mesh on the interval.

Clearly, in general, we can't find analytical solution of integrals on right side in (9). Therefore, we will approximate function *f(x,y(x))* on the interval $\left[x_i, x_{i+1}\right]$ as follows:

$$\bar{f}\left(x, y(x)\right) = f\left(x_i, y(x_i)\right) \quad \left(x \in [x_i, x_{i+1}]\right) \qquad \left(i = 0,1,\ldots,\ N-1\right).$$

Now, from (9) and after assortiment, we have the new difference scheme

$$a_i \bar{y}_{i-1} - c_i \bar{y}_i + b_i \bar{y}^{I}_{i+1} = \frac{1}{\varepsilon^2}\left[\left(\int_{x_{i-1}}^{x_i} u^{II}_{i-1}\, ds\right) f(x_{i-1}, \bar{y}_{i-1}) + \left(\int_{x_i}^{x_{i+1}} u^{I}_{i}\, ds\right) f(x_i, \bar{y}_i) \right] \quad \left(i = 1,2,\ldots,\ N-1\right),$$

where $\bar{y}_i$ $\left(i = 1,2,\ldots,\ N-1\right)\left(i = 1,2,\ldots,\ N-1\right)$ are approximated values of solution *y(x)* which is solution of the problem (1) – (3) in points x_i $\left(i = 1,2,\ldots,\ N-1\right)$.

From the last difference scheme we have

$$a_i \bar{y}_{i-1} - c_i \bar{y}_i + a_{i+1} \bar{y}_{i+1} = \frac{d_i - a_i}{p} f(x_{i-1}, \bar{y}_{i-1}) + \frac{d_{i+1} - a_{i+1}}{p} f(x_i, \bar{y}_i) \qquad \left(i = 1,\ldots,\ N-1\right), \tag{10}$$

$y_0 = 0$; $y_N = 0$, where $d_i = \dfrac{\beta}{\tanh\left(\beta h_{i-1}\right)}$.

Note that, matrix of the left side of the system of equations (10) is threediagonal symmetric invertible, which with condition (3) gives us uniqueness of solution of the system (10).

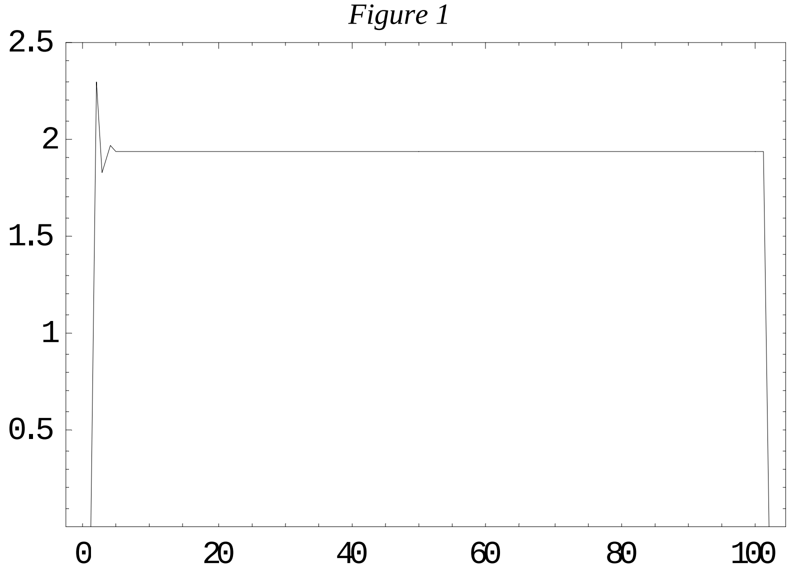
3. NUMERICAL EXPERIMENT

Using difference scheme (10) we will consider and find approximated solutions of nonlinear reaction-diffusion problem:

$$\varepsilon\, y'' = y^3 - 8 \quad \text{on } (0,1) \tag{11}$$
$$y(0) = y(1) = 0.$$

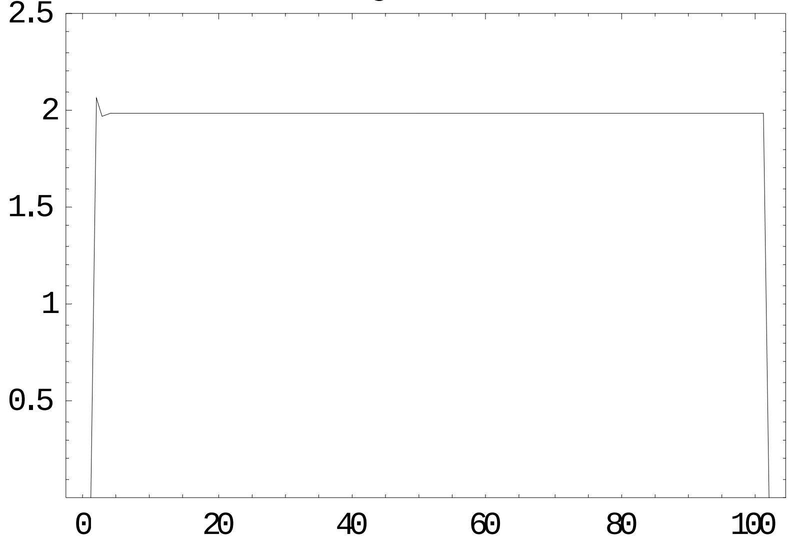
Using differece scheme (10) , *Figure 1* shows solution $\bar{y}(x)$ of our problem for the equdistance mesh of the interval $[0,1]$, ($\varepsilon^2 = 0.0001$, $N = 100$).

Figure 1



Using differece scheme (10) , *Figure 2* shows solution $\bar{y}(x)$ of our problem for the non-equdistance mesh of the interval $[0,1]$, ($\varepsilon^2 = 0.0001$, $N = 100$). The non-equdistance mesh is constructed to give more points in a boundary lajer.

Figure 2



We can see from the graphic that the function $\bar{y}(x)$ is in "neighbourhood" of the constant function $y(x)=2$ for $x \in (0,1)$.

Remark: All calculations in this paper were done using program package MATEMATIKA 5.0.

4. REFERENCES

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