11th International Research/Expert Conference
"Trends in the Development of Machinery and Associated Technology"
TMT 2007, Hammamet, Tunisia, 05-09 September, 2007.

A MATHEMATICAL APPROACH OF THE REYNOLDS EQUATION FOR AXIAL TILTING PAD BEARINGS

Ioan C. Țarcă
Voichița I. Hule
University of Oradea
1 University str., Oradea
Romania

ABSTRACT
Significant differences were noticed between the theoretical optimum pad slope [2], [3] and experimental values prescribed in catalogues [6], [7]. This paper tries to reveal the main cause of these differences and proposes a mathematical method that allows the calculus of the pressure in the lubricant film, based on Reynolds equation. Helicoidally shaped thrust pad were taken into account due to the simplicity of the lubricant film shape.
Keywords: tilting pad bearing, Reynolds equation, helicoidally shaped pad.

1. THE MATHEMATICAL MODEL
Noticeable simplifications in Reynolds' equation integration occur during the equivalence of the curve-shaped pad with a rectangular one. But these simplifications are accompanied by a distortion of the real phenomenon as follows: 1) due to the circular shape of the pad tangential speeds presents a radial gradient ($U=r\omega$, where r is the radius coordinate of a point on the pad and ω is the angular velocity of the shaft); 2) inertia effects inside the lubricant film are neglected and one of the main effect is the fact that the pressure inside the film decreases due to the lubricant leakage at the exterior boundary of the film in radial direction, which is proportional to angular velocity; 3) mathematical model in the theory is attained analyzing the fluid flow by means of Cartesian coordinates after which the axial bearing model is achieved through cylindrical coordinates transformation.
The authors proposes the study of the pressure, friction and inertia forces distribution inside the lubricant film using a model which takes into account an infinitesimal cylinder sector $rd\theta$, dr, dz, separated from a cylinder having the axial coordinate Oz_1 (figure 1).

2. GOVERNING EQUATION
Mainly, simplifications considered in the paper are the same used relative to the lubricant flow in bearings, except for those regarding the curvature of the film, i.e. inertia forces and the variation of tangential speed with radius. In these assumptions, Reynolds equation can be written as:

$$\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{h^3}{\eta}\frac{\partial p}{\partial\theta}\right)+\frac{\partial}{\partial r}\left(\frac{h^3}{\eta}\left(\frac{\partial p}{\partial r}-\frac{p}{r}-\rho\cdot\omega^2\cdot r\right)\right)=6\cdot\omega\frac{\partial h}{\partial\theta} \qquad (1)$$

Considering that the surface of the pad is helicoidally, it is obvious that the height of the film h is constant in radial direction, given by:

$$h=(1+m)h_2-a\cdot\theta \qquad (2)$$

where $a=\frac{h_1-h_2}{\Delta\theta}=\frac{mh_2}{\Delta\theta}=r\cdot tg\alpha=const.$ and $m=\frac{h_1}{h_2}$, according to figure 2.

Equation 1 was studied in respect with its coordinates, θ and r.

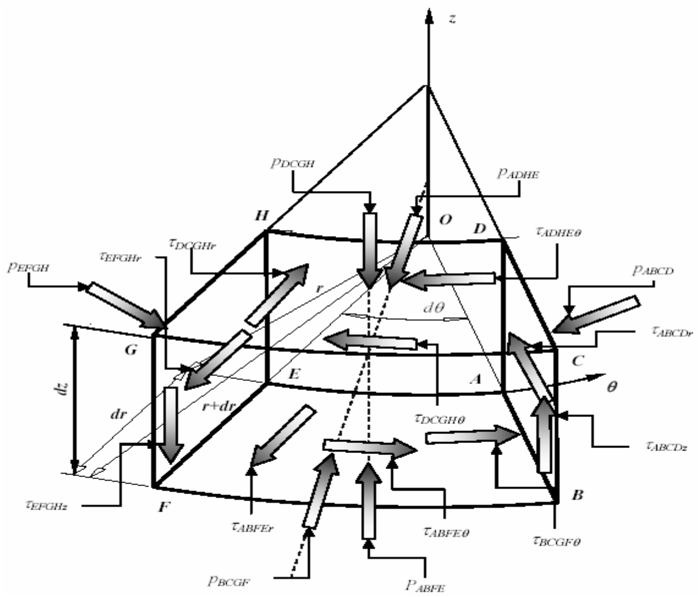

Figure 1. Forces distribution for cylindrical shaped fluid film

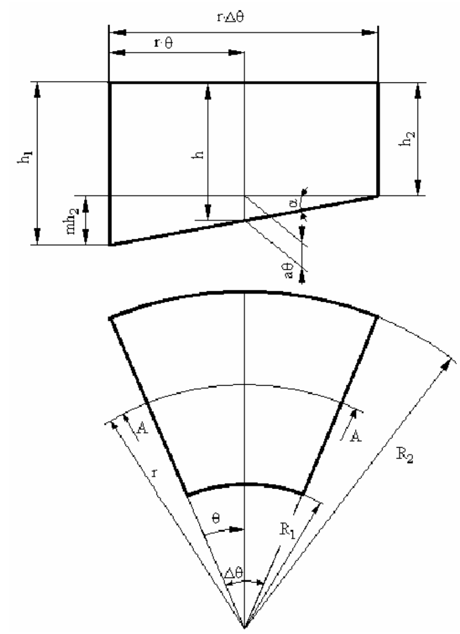

Figure 2. Notations for axial bearing

Case 1: for a given radius, *r*=const., so $\frac{\partial p}{\partial r}=0$. Also $\frac{\partial h}{\partial r}=0$ and $\frac{\partial h}{\partial \theta}=-a$.

In these assumptions eq.(1) becomes:

$$h^3\left(\frac{\partial^2}{\partial h^2}p(h)\right)+3h^2\left(\frac{\partial}{\partial h}p(h)\right)+\frac{h^3}{a^2}p(h)=\frac{bh^3}{a^2}-\frac{c}{a} \qquad (3)$$

where $b=\rho\cdot\omega^2\cdot r^2$ and $c=6\cdot\eta\cdot\omega\cdot r^2$

The solution of the equation (3) is:

$$p(h)=-\frac{1}{2}\frac{\pi\left(\int\frac{(bh^3-ca)BesselY\left(1,\frac{h}{a}\right)}{h}dh\cdot BesselJ\left(1,\frac{h}{a}\right)-\int\frac{(bh^3-ca)BesselJ\left(1,\frac{h}{a}\right)}{h}dh\cdot BesselY\left(1,\frac{h}{a}\right)\right)}{ha^2}+ \qquad (4)$$

$$+\frac{C1\cdot BesselJ\left(1,\frac{h}{a}\right)}{h}+\frac{C2\cdot BesselY\left(1,\frac{h}{a}\right)}{h}$$

where $BesselJ\left(1,\frac{h}{a}\right)$ and $BesselY\left(1,\frac{h}{a}\right)$ are first order Bessel functions for eq.(3). Development of Bessel functions yields to:

$$BesselJ\left(1,\frac{h}{a}\right)=\frac{1}{2a}h-\frac{1}{16a^3}h^3+\frac{1}{384a^5}h^5+O\left(h^6\right)$$

$$BesselY\left(1,\frac{h}{a}\right)=-2\frac{a}{\pi}\frac{1}{h}+\left(\frac{\ln\frac{h}{2a}+\gamma-\frac{1}{2}}{\pi a}\right)h+\left(-\frac{1}{8}\frac{\ln\frac{h}{2a}+\gamma-\frac{5}{4}}{\pi a^3}\right)h^3+\left(\frac{1}{192}\frac{\ln\frac{h}{2a}+\gamma-\frac{5}{3}}{\pi a^5}\right)h^5+O\left(h^6\right) \qquad (5)$$

where $\gamma=\lim_{n\to\infty}\left(\ln n\sum_{i=1}^{n}\frac{1}{i}\right)$ represents the Euler constant, being equal to 0.5772156649 (approx. value).

Keeping only significant members of the series (neglecting terms corresponding to h^3 and h^5), and substituting $\varepsilon = \frac{h}{h_2}$, and notifying that the integration constants C1 and C2 are computed using the following boundary conditions: $p(m+1) = p_a$ and $p(1) = p_a$, the lubricant pressure inside gap in circumferential direction is given by:

$$p(\varepsilon) = \frac{1}{64}\frac{\Delta\theta^4 b}{m^4}\varepsilon^4 + \frac{1}{8}\frac{\Delta\theta^2 b h_2^2}{m^2}\varepsilon^2 - \frac{1}{4}\frac{\Delta\theta^3 c}{h_2^2 m^3}\varepsilon - A\ln\left(\frac{\Delta\theta\varepsilon}{2m}\right) + \frac{\Delta\theta c}{h_2^2 m}\frac{1}{\varepsilon} - B\frac{1}{\varepsilon^2} + C \qquad (6)$$

where A, B and C are coefficients. Maximum pressure in circumferential direction develops at the point where $\frac{\partial p}{\partial \theta} = 0$ or $\frac{\partial p}{\partial h} = 0$, as the solution of the equation $\frac{\partial p(\varepsilon)}{\partial \varepsilon} = 0$ (a fourth degree equation in ε) which has a unique real solution, that is:

$$\begin{aligned}&\left(4\Delta\theta^3 b h_2^2 (m+1)^2 \ln(m+1) + 8\Delta\theta b h_2^2 m^3 (m+2)\right)\varepsilon^4 - 16\left(\Delta\theta^2 c m (m+1)^2 \ln(m+1) + 2m^4 c (m+2)\right)\varepsilon - \\ &- h_2^2 m \Delta\theta (m+1)^2 (m+2)\left(\Delta\theta^2\left(m^2 + 2m + 2\right) + 8m^2\right)b + 16m^2 (m+1)\left(\Delta\theta^2 (m+1) + 4m^2\right)c = 0\end{aligned} \qquad (7)$$

The other two solutions are imaginary $\left(\pm 2\frac{m}{\Delta\theta}i\right)$, and they are irrelevant. The solution of the eq. (7) should be searched inside the range $(1, m+1)$, and it can be computed through numerical methods.

Case 2: for a given angle $\theta = const.$, pressure depends only on radius, i.e. $\partial p/\partial\theta = 0$. In these assumptions eq.(1) becomes:

$$\frac{\partial^2}{\partial r^2}p(r) - \frac{1}{r}\frac{\partial}{\partial r}p(r) + \frac{p(r)}{r^2} = b - c \qquad (8)$$

where $b = \rho\omega^2$, and $c = \frac{6\eta\omega a}{h^3}$.

The solution of eq.(8) is:

$$p(r) = (b-c)r^2 + C1r + C2r\ln(r) \qquad (9)$$

Constants $C1$ and $C2$ are computed using the following boundary conditions: $p(R_1) = p_a$ and $p(R_2) = p_a$

Pressure in radial direction should therefore be:

$$\begin{aligned}p(r) = &\left(\rho\omega^2 - \frac{6\eta\omega a}{\left(h_2 + mh_2(\Delta\theta/\theta - 1)\right)^3}\right)r^2 + \frac{\left(\left(\rho\omega^2 - \frac{6\eta\omega a}{h^3}\right)R_1R_2 + p_a\right)(R_1 - R_2)}{R_1R_2\ln(R_2/R_1)}r\ln(r) + \\ &\frac{\left(\rho\omega^2 - \frac{6\eta\omega a}{h_2 + mh_2(\Delta\theta/\theta - 1)^3}\right)R_1R_2(R_2\ln R_1 - R_1\ln R_2) - p_a(R_1\ln R_1 - R_2\ln R_2)}{R_1R_2\ln(R_2/R_1)}r\end{aligned} \qquad (10)$$

Maximum pressure develops at the point where $\frac{\partial p}{\partial r} = 0$; the solution of the equation is:

$$r\big|_{p(r)\max} = e^{\left(-\left[\mathrm{LambertW}\left(-2\frac{\left(\rho\omega^2 - \frac{6\eta\omega a}{h_2 + mh_2\left(\frac{\Delta\theta - \theta}{\theta}\right)^3}\right)e^{\left(-\frac{C1+C2}{C2}\right)}}{C2}\right)C2 + C1 + C2\right]\Big/C2\right)} \qquad (11)$$

3. OPTIMUM CONDITIONS

Two conditions were considered to achieve the optimum for the thrust bearing:

Load maximization

Friction minimization

The first assessment condition means that for given load conditions (*W*), velocity (*V*), and dimensions $R_1, R_2, \Delta\theta$, maximum pressure should be obtained modifying the slope of the pad *m*. The second assessment is equivalent to the minimization of the friction coefficient, μ [4]:

$$\mu(m) = \frac{mh_2\left((m+2)\ln(m+1)+2m\right)}{\Delta\theta(R_1+R_2)\left((m+2)\ln(m+1)+2m\right)} \qquad (12)$$

Minimum value for this coefficient occurs when $\partial\mu(m)/m = 0$, yielding to the following equation in *m*:

$$(m+1)(m+2)^2 \ln(m+1)^2 + 8m(m+1)\ln(m+1) - 4m^2(2m+3) = 0 \qquad (13)$$

The solution of this equation is $m = 2.80448$.

4. CONCLUSIONS

Analyzing the influence of pad slope *m* on pressure, it can be noticed that significant differences occurs between the classical model of Reynolds equation and the equation (1). Sensible higher values were noticed considering eq.(1), along with the fact that the slope for the diagrams were more acute, so that it can be concluded that in the case of the authors' model the point of maximum load is very sensible relative to pad slope. Maximum pressure is achieved for m=1.4 in classical assumptions, while considering the real shape of the pad, the value for *m* is between 1 and 1.1, in good correspondence with producers' catalogues [6], [7].

Dimensionless coordinate ε has lesser influence on pressure than the slope *m* in circumferential direction, little higher values for pressure being notified in case of considering circular shape for pad.

In radial direction differences were insignificant, both for *m* and *r* parameters.

Considering the minimization of friction forces together with the maximization of the load, it was noticed that an optimum value for *m* is included in the range [1, 2.804], the value 1 corresponding to maximum load, and 2.804 for minimum power loss through friction.

5. REFERENCES

[1] Carafoli, E., Constantinescu, V. N., Dinamica fluidelor incompresibile, Bucureşti, Editura Academiei, 1981

[2] Chişiu, Al., Mătieşan, D., Mădărăşan, T., Pop D., Organe de maşini, Bucureşti, Editura Didactică şi Pedagogică, 1981

[3] Constantinescu, V. N., Nica, Al., Pascovici, M. D., Ceptureanu, Gh., Nedelcu, Şt., Lagăre cu alunecare, Bucureşti, Editura Tehnică, 1980

[4] Halling, J., Principals of Tribology, The Mac Millan Press, 1975

[5] Lang, O. R., Steihilper, W., Gletilager. Berechnung und Konstruktion von Gleitlagern mit konstanter und zeitlich veränderlicher Belastung, Berlin, Heidelberg, New York, Springer-Verlag, 1978

[6] www.waukbearing.com, Tilting Pad Radial Bearings. Metric Range. Catalog.

[7] www.kingsbury.com, A General Guide to the Principles, Operation and Troubleshooting of Hydrodynamic Bearings

[8] Ţarcă, I., Approach on the design optimization of axial bearings, Ph.D. Thesis, Polytechnic University of Cluj Napoca, 2004

[9] Ţarcă, I., Ţarcă, R., Aspects regarding optimization for hydrodynamic thrust pad bearings, PRASIC 2006, 9-10 November, Braşov, ISBN (10)973-635-824-0; (13)978-973-635-824-1