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ROBUST CONTROL OF UNCERTAIN TIME-DELAY SYSTEMS

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ABSTRACT
The time delay in transport of material and energy is an inseparable part of many real industrial processes. Unfortunately, the presence of time-delay terms itself almost always brings problems in control and, quite naturally, the situation is even more complicated if this delay is uncertain. The source of the uncertainty can arise e.g. from simplifications during mathematical modelling, changes in physical parameters or various operational conditions. The main aim of the contribution is to show capabilities of robust algorithms designed via an algebraic approach in control of systems with parametrically uncertain time delay. The robust stability of the closed loop is graphically analyzed using the value set concept and the zero exclusion condition.
Keywords: robust control, time-delay systems, parametric uncertainty

1. INTRODUCTION
The time-delay systems have been the envy of control researchers and engineers attention for decades. The ground of this interest lies in the common presence of various delays in real control loops which, quite naturally, has brought the requirement of adequate control algorithms. Unluckily, time delay substantially deteriorates the control conditions and, moreover, if the delay is uncertain, the issue of quality control becomes even more complex.
An effective solution of this problem is represented by the utilization of continuous-time robust controllers. A possible approach to robust control design is represented by fractional approach developed in [9, 4] and utilized (not only) for time-delay systems e.g. in [5, 6, 7]. This technique is based on general solutions of Diophantine equations in the ring of proper and stable rational functions (R_{PS}), Youla-Kučera parameterization and conditions of divisibility and it supposes approximation of time-delay term.
This contribution is focused on control of single-input single-output systems with parametrically uncertain time delay using the regulators computed via proposed control design method. The robust stability of closed-loop characteristic quasipolynomials is investigated through the value set concept in combination with the zero exclusion condition. The design process and robust stability analysis is demonstrated on the example of first order uncertain dominant time-delay plant and PID controller.

2. DESCRIPTION OF THE UNCERTAIN TIME-DELAY PLANT
The controlled process is assumed to be given by first order time-delay transfer function:

$$G(s,\Theta) = \frac{K}{Ts+1} e^{-\Theta s} \tag{1}$$

where gain $K = 5$, time constant $T = 10$ and time-delay can vary within given interval $\Theta \in \langle 5, 35 \rangle$. The delay in the nominal system, used for control design, is fixed to $\Theta = 20$. Furthermore, for the sake of applicability of this transfer function in algebraic synthesis method, it is necessary to approximate the time-delay term. The well-known and popular Padé approximation can be employed for this purpose. The use of its first order (linear) version leads to the function:

$$G_N(s) = \frac{5}{10s+1}e^{-20s} \approx \frac{5(1-10s)}{(10s+1)(1+10s)} = \frac{-50s+5}{100s^2+20s+1} = \frac{-0.5s+0.05}{s^2+0.2s+0.01} \tag{2}$$

3. CONTROLLER DESIGN

Suppose the conventional feedback control system with plant $G(s) = b(s)/a(s)$ and controller $C(s) = q(s)/p(s)$. The fractional approach developed by Vidyasagar [9] and Kučera [4] and refined in [5, 6, 7] is based on general solutions of Diophantine equations in R_{PS}. The conversion from common polynomial representation to the R_{PS} notation can take the form:

$$G(s) = \frac{b(s)}{a(s)} = \frac{\dfrac{b(s)}{(s+m)^n}}{\dfrac{a(s)}{(s+m)^n}} = \frac{B(s)}{A(s)} \tag{3}$$

where $n = \max\{\deg(a), \deg(b)\}$ and $m > 0$. The parameter *m*, entering into synthesis process, represents the tool which will be subsequently used for influencing the properties of closed-loop control responses.

The first and definitely the most important requirement is to ensure the stability of control loop. Stabilizing controllers are given by ratio (no longer with accentuated complex variable *s*):

$$\frac{Q}{P} = \frac{Q_0 - AT}{P_0 + BT} \tag{4}$$

where *T* is free in R_{PS}, $P_0 + BT \neq 0$ and P_0, Q_0 is some particular solution of Diophantine equation:

$$AP + BQ = 1 \tag{5}$$

The formula (4) says that there exists either infinite amount of stabilizing regulators or none and it is called Youla-Kučera parameterization of controllers.

Another important property is the convergency of tracking error *e* to zero. Working on an assumption that no disturbances affect the control system and reference signal is given by $w = G_w/F_w$, it follows:

$$e = \frac{AP}{AP+BQ}\frac{G_w}{F_w} \tag{6}$$

Algebraic analysis of (6) results in fact that for zero tracking error, the expression F_w must divide product AP. The divisibility in R_{PS} is defined somewhat abstractly: x/y divides $\tilde{x}/\tilde{y}$ if and only if all zeros of x/y located in right complex half plane (including imaginary axis and infinity) are also zeros of $\tilde{x}/\tilde{y}$.

The control design is illustrated on the following example. The last form of function (2) determines the required case of second order controlled plant, generally:

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \tag{7}$$

Then, the basic Diophantine equation (5) can be expressed as:

$$\frac{s^2 + a_1 s + a_0}{(s+m)^2}\frac{p_1 s + p_0}{s+m} + \frac{b_1 s + b_0}{(s+m)^2}\frac{q_1 s + q_0}{s+m} = 1 \tag{8}$$

Its particular solution is:

$$p_1 = 1; \quad p_0 = \frac{3m^2 b_0 b_1 - a_0 b_0 b_1 - 3mb_0{}^2 + a_1 b_0{}^2 - b_1{}^2 m^3}{a_1 b_0 b_1 - b_0{}^2 - a_0 b_1{}^2}; \quad q_1 = \frac{3m - a_1 - p_0}{b_1}; \quad q_0 = \frac{m^3 - a_0 p_0}{b_0} \tag{9}$$

Youla-Kučera parameterization purveys all solutions of the equation (8):

$$\begin{aligned} P &= P_0 + BT = \frac{s + p_0}{s+m} + \frac{b_1 s + b_0}{(s+m)^2} T \\ Q &= Q_0 - AT = \frac{q_1 s + q_0}{s+m} - \frac{s^2 + a_1 s + a_0}{(s+m)^2} T \end{aligned} \tag{10}$$

Suppose the step changes in reference signal and thus $F_w = s/(s+m)$. Now, it is necessary to choose such controller from the set (10) in order to F_w divides AP. Hence, it has to be found $T = t_0$ so that term *s* can be separated from the numerator of *P*. After certain adjustment it follows that complying t_0 is the one and only, scilicet $t_0 = -p_0 m / b_0$. By its substitution into (10), the numerator and denominator of the controller, which will not only stabilize the controlled plant in closed-loop system but it will also guarantee the asymptotic tracking of the reference signal, are obtained:

$$\begin{aligned} P &= \frac{s^2 + s\left(p_0 + m - p_0 m \dfrac{b_1}{b_0}\right)}{(s+m)^2} = \frac{s^2 + \tilde{p}_1 s}{(s+m)^2} \\ Q &= \frac{s^2\left(q_1 + \dfrac{p_0 m}{b_0}\right) + s\left(q_0 + q_1 m + a_1 \dfrac{p_0 m}{b_0}\right) + a_0 \dfrac{p_0 m}{b_0} + q_0 m}{(s+m)^2} = \frac{\tilde{q}_2 s^2 + \tilde{q}_1 s + \tilde{q}_0}{(s+m)^2} \end{aligned} \tag{11}$$

Thus, the transfer function of final PID-like feedback controller is in the form:

$$\frac{Q}{P} = \frac{\tilde{q}_2 s^2 + \tilde{q}_1 s + \tilde{q}_0}{s(s + \tilde{p}_1)} \tag{12}$$

with parameters:

$$\tilde{p}_1 = p_0 + m - p_0 m \frac{b_1}{b_0}; \quad \tilde{q}_2 = q_1 + \frac{p_0 m}{b_0}; \quad \tilde{q}_1 = q_0 + q_1 m + a_1 \frac{p_0 m}{b_0}; \quad \tilde{q}_0 = q_0 m + a_0 \frac{p_0 m}{b_0} \tag{13}$$

4. ROBUST STABILITY ANALYSIS AND CONTROL SIMULATIONS

Assume the approximated nominal plant (2) and the tuning parameter $m = 0.06$. Utilization of relations from the previous chapter leads to the controller:

$$\frac{Q}{P} = \frac{\tilde{q}_2 s^2 + \tilde{q}_1 s + \tilde{q}_0}{s(s + \tilde{p}_1)} = \frac{0.04768 s^2 + 0.007104 s + 0.0002592}{s(s + 0.06384)} \tag{14}$$

The question is, if this regulator stabilizes the whole family of true systems (1) for all possible values of uncertain time-delay, i.e. if the control system is robustly stable. This problem can be solved using the graphical test, which relies on depiction of the closed-loop characteristic (quasi)polynomial value sets and application of the zero exclusion condition. An array of information about robustness of systems with parametric uncertainty and related topics can be found in [1, 2, 3, 8]. The family of closed-loop characteristic quasipolynomials is given by:

$$p_{CL}(s,\Theta) = (Ts+1)\left(s^2 + \tilde{p}_1\right) + Ke^{-\Theta s}\left(\tilde{q}_2 s^2 + \tilde{q}_1 s + \tilde{q}_0\right); \qquad \Theta \in \langle 5, 35 \rangle \tag{15}$$

Roughly speaking, the value set for one fixed frequency ω can be obtained by substitution of *s* for $j\omega$ in the family (15) and letting the delay Θ range over the prescribed set. The figure 1 shows such value sets plotted in complex plane for $\omega = \langle 0, 0.15 \rangle$ with step 0.001. The quasipolynomial (15) and thus also whole control system is robustly stable, because the family has a stable member and the origin of the complex plane is excluded from the value sets. The robust stability is confirmed also by figure 2, where final set of closed-loop control responses of the plant (1) with the regulator (14) can be seen. This simulation was obtained under conditions as follows: The 301 "representative" systems were chosen by sampling the time-delay term from the set of uncertain plants (step 0.1, i.e. 301 systems for simulation + highlighted nominal one). Furthermore, the step load disturbance -0.1 affects the input of the controlled system during the last third of simulation time.

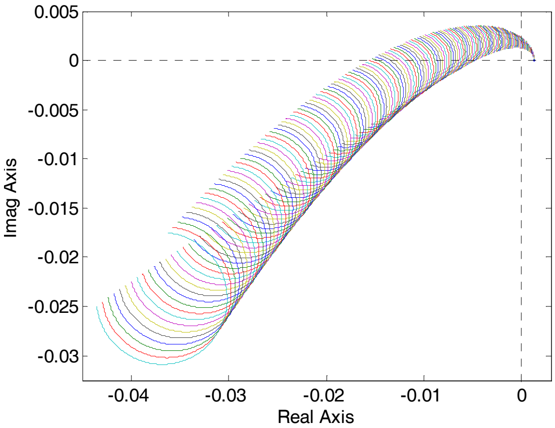

Figure 1. The value sets of uncertain characteristic quasipolynomial (15)

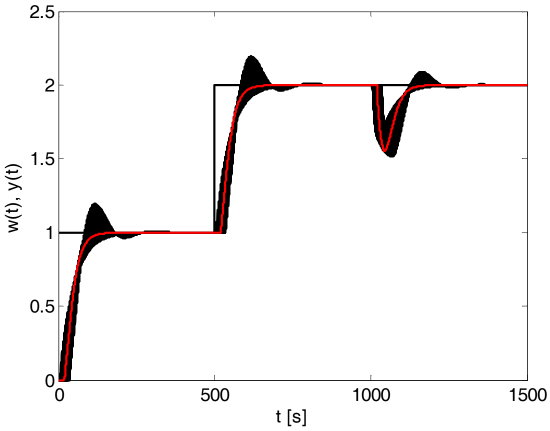

Figure 2. The set of control responses

5. CONCLUSIONS

The paper has presented the capabilities of continuous-time robust algorithms designed through an algebraic approach in control of systems with parametrically uncertain time delay. The essential theoretical foundations are followed by an illustrative example with dominant time-delay plant.

6. ACKNOWLEDGEMENT

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