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DIFFERENCE SCHEME for SEMILINEAR REACTION-DIFFUZION PROBLEM of ELIPTIC TYPE

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ABSTRACT
In this paper we consider semilinear elliptic Dirichlet`s boundary value problem with small parameter, well known as singularly perturbed semilinear reaction-diffusion problem. Using theory of projection-mesh methods, precisely using the Galerkin method with natural choice of test function, the given boundary problem is discretized and we get a discrete analog non-linear system. Solving the non-linear difference scheme, we come to the approximate solution of the problem.
Keywords: semi-linear elliptic problem, difference scheme, semi-linear reaction-difusion problem

1. INTRODUCTION
In rectangular domain $\Omega = \{(x, y) \in R^2 : 0 < x < l_1,\ 0 < y < l_2\}$,

consider semilinear elliptic equation

$$\varepsilon^2 \Delta u(x, y) = F(x, y; u) \qquad (\varepsilon \in (0,1]\ ) \tag{1}$$

where $F(x, y; u)$ is sufficiently regular function, which has strictly positive partial derivative with respect to u,

$$F_u = \frac{\partial F(x, y; u)}{\partial u} \geq m > 0 \qquad (m = const.)\ . \tag{2}$$

Propouse Dirichlet's boundary condition : $u(x, y)\big|_G = 0$, (G bound area Ω).

(3)

We call problem (1) – (3) semilinear elliptic Dirichlet's boundary problem. Estimate solution of this problem is given in [3]. Notice that analog one-dimensional problem is considered in [2] and will be basic start point for discretization of problem (1) – (3). The aim of mentioned discretization is getting difference sheme which will, uniformly converge to exact solution of problem (1)–(3) (in points of our mesh), with respect to perturbed parameter ε.

2. DISCRETIZATION OF THE PROBLEM
Define mesh $\overline{\omega}_h = \overline{\omega}_h^1 \times \overline{\omega}_h^2$, on set $\overline{\Omega}$, where

$$\overline{\omega}_h^1 = \{\ x_i\ :\ i = 0,1,..., N_1\ ; x_0 = 0, x_{N_1} = l_1\ \},$$

$$\overline{\omega}_h^2 = \{\ y_j : j = 0,1,...,N_2 ; y_0 = 0, y_{N_2} = l_2\ \}.$$

Let $G_h = \overline{\omega}_h \cap G$ and $\omega_h = \overline{\omega}_h \cap \Omega$. Thus, $\overline{\omega}_h = \omega_h \cup G_h$, where sets in last union are disjoint.

Let $\psi(t,w) = f(t,w) - \gamma w \quad (\gamma = const. > 0)$. Now, we can consider next boundary problem

$$L_\varepsilon w \equiv \varepsilon^2 w''(t) - \gamma\ w(t) = \psi\ (t,w) \text{ on } (0,1)$$

$$w(0) = 0 \quad ; \quad w(l) = 0, \tag{4}$$

on given mesh

$$\{t_i : i = 0,1,...,N ; t_0 = 0 ; t_N = l\}.$$

Using results from $[2]$, we have

$$(T_w)_i = a_i\, w_{i-1} - c_i\, w_i + a_{i+1}\, w_{i+1} = \varepsilon^{-2}\left[\int_{t_{i-1}}^{t_i} u_{i-1}^{II}(s)\psi\ (s,w)ds + \int_{t_i}^{t_{i+1}} u_i^I(s)\ \psi(s,w)ds\right],$$

$$i = 1,2,...,\ N-1 \ ; \quad w_0 = w_N = 0\ , \tag{5}$$

where

$$w_i = w(t_i) \quad ; \quad a_i = \frac{\beta}{\sinh(\beta\,\Delta\ t_{i-1})} \quad ; \quad c_i = \frac{\beta}{\tanh(\beta\,\Delta\ t_{i-1})} + \frac{\beta}{\tanh(\beta\,\Delta\ t_i)} \quad ; \quad \beta = \frac{\sqrt{\gamma}}{\varepsilon}.$$

If we approximate right side in (5), by replacing function $\psi(t,w(t))$ on $[t_i, t_{i+1}]$ with its value in point t_i, we get next difference sheme

$$a_i\, \overline{w}_{i-1} - c_i\, \overline{w}_i + a_{i+1}\, \overline{w}_{i+1} = \frac{\Delta\ c_i}{\gamma}\psi\ (t_i, \overline{w}_i)\text{ , where}$$

$$\frac{\Delta\ c_i}{\gamma} = \frac{1}{\varepsilon^2}\left[\int_{t_{i-1}}^{t_i} u_{i-1}^{II}(s)\ ds + \int_{t_i}^{t_{i+1}} u_i^I(s)\ ds\right].$$

Above difference sheme can be written as

$$(T_h\overline{w})_i \equiv \frac{\gamma}{\Delta\ c_i}\left(a_{i+1}(\overline{w}_{i+1} - \overline{w}_i) - a_i(\overline{w}_i - \overline{w}_{i-1})\right) = f(t_i, \overline{w}_i)$$

$$i = 1,2,...,\ N-1 \ ;\ w_0 = w_N = 0\ . \tag{6}$$

On the line $y = y_j \quad (j = 0,1,...,N_2 - 1)$, equation (5) becomes

$$\left(T_u^{(1)}\right)_{i,j} \equiv a_i^{(1)} u_{i-1,j} - c_i^{(1)} u_{i,j} + a_i^{(1)} u_{i+1,j} = \varepsilon^{-2}\left[\int_{t_{i-1}}^{t_i} u_{i-1}^{II(x)}(s)\psi_j^{(1)}(s)ds + \int_{t_i}^{t_{i+1}} u_i^{I(x)}(s)\psi_j^{(1)}(s)\ ds\right],$$

$$i = 1,2,...,N_1 - 1 ;\ u_{0,j} = u_{N_1,j} = 0\ , \tag{7}$$

where

$$\psi_j^{(1)}(s) = F\ (s, y_j\ ; u(s,y_j)) - \gamma_1 u(s,y_j) - \varepsilon^2\frac{\partial^2 u(s,y_i)}{\partial y^2}\ ,$$

$u_{i,j} = u(x_i, y_j) \quad ; \quad \gamma_1 = const. > 0$, and $u_i^{I(x)}$, $u_i^{II(x)}$ are solutions of boundary problems

$$\varepsilon^2 \frac{\partial^2 u_i^I(x, y_j)}{\partial x^2} - \gamma_1 u_i^I(x, y_j) = 0 \quad ; \quad u_i^{I(x)}(x_i) = 1 \ ; \ u_i^{II(x)}(x_{i+1}) = 0 ,$$

$$\varepsilon^2 \frac{\partial^2 u_i^{II}(x, y_j)}{\partial x^2} - \gamma_1 u_i^{II}(x, y_j) = 0 \quad ; \quad u_i^{II(x)}(x_i) = 0 \, ; u_i^{II(x)}(x_{i+1}) = 1 .$$

Similarly, on the line $x = x_i \quad (i = 0,1,..., N_1 - 1)$, equation (5) becomes

$$(T_u^{(2)})_{i,j} = a_j^{(2)} u_{i,j-1} - c_j^{(2)} u_{i,j} + a_{j+1}^{(2)} u_{i,j+1} = \varepsilon^{-2} \left[\int_{y_{j-1}}^{y_j} u_{j-1}^{II(y)}(s)\psi_i^{(2)}(s)ds + \int_{y_j}^{y_{j+1}} u_j^{I(y)}(s)\psi_i^{(2)}(s)ds \right],$$

$$j = 1,2,..., N_2 - 1 ; \ u_{i,0} = u_{i,N_2} = 0 \ , \tag{8}$$

where $\psi_i^{(2)}(s) = F(x_i, s; u(x_i, s)) - \gamma_2 u(x_i, s) - \varepsilon^2 \frac{\partial^2 u(x_i s)}{\partial x^2}$,

$\gamma_2 = const. > 0$, and $u_j^{I(y)}$, $u_j^{II(y)}$ are solutions of boundary problem

$$\varepsilon^2 \frac{\partial^2 u_j^I(x_i, y)}{\partial y^2} - \gamma_{21} u_j^I(x_i, y) = 0 \quad ; \quad u_j^{I(y)}(y_j) = 1 \, ; u_j^{I(y)}(y_{j+1}) = 0 ,$$

$$\varepsilon^2 \frac{\partial^2 u_j^{II}(x_i, y)}{\partial y^2} - \gamma_2 u_j^{II}(x_i, y) = 0 \quad ; \quad u_j^{II(y)}(y_j) = 0 \, ; u_j^{II(j)}(y_{j+1}) = 1 .$$

Obviously, we can represent functions $\psi_j^{(1)}(s)$ and $\psi_i^{(2)}(s)$ as

$$\psi_j^{(1)}(s) = \psi_j^{(1)}(x_i) + [\ \psi_j^{(1)}(s) - \psi_j^{(1)}(x_i)\ ] = \psi_j^{(1)}(x_i) + \int_{y_j}^{s} d\psi_j^{(1)} ,$$

$$\psi_i^{(2)}(s) = \psi_j^{(2)}(y_j) + [\ \psi_i^{(2)}(s) - \psi_i^{(2)}(y_j)\ ] = \psi_i^{(2)}(y_j) + \int_{y_j}^{s} d\psi_i^{(2)} .$$

Now, equations (7) and (8) can be write as

$$T_h^{(1)} u = F(x, y; u) - \varepsilon^2 \frac{\partial^2 u}{\partial y^2} + Y^{(1)}(x, y; \psi^{(1)}) , \tag{9a}$$

$$T_h^{(2)} u = F(x, y; u) - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + Y^{(2)}(x, y; \psi^{(2)}) , \tag{9b}$$

for $(x, y) \in \omega_h$, where

$$\Upsilon^{(1)}(x_i . y_j ; \psi^{(1)}) = \frac{\gamma_1}{\varepsilon^2 \Delta c_i^{(1)}} [\ \int_{x_{i-1}}^{x_i} u_{i-1}^{II(x)}(s)(\int_{x_i}^{s} d\psi_j^{(1)}) ds + \int_{x_i}^{x_{i+1}} u_i^{I(x)}(s)(\int_{x_i}^{s} d\psi_j^{(1)}) ds\ ] ,$$

$$\Upsilon^{(2)}(x_i . y_j ; \psi^{(2)}) = \frac{\gamma_2}{\varepsilon^2 \Delta c_j^{(1)}} [\ \int_{y_{j-1}}^{y_j} u_{j-1}^{II(y)}(s)(\int_{y_j}^{s} d\psi_j^{(2)}) ds + \int_{y_j}^{y_{j+1}} u_j^{I(y)}(s)(\int_{y_j}^{s} d\psi_j^{(2)}) ds\ ] .$$

Operators $T_h^{(1)}$ and $T_h^{(2)}$ are defined analogusly as in (6). If now, for $(x, y) \in \omega_h$, by adding equations (9a) and (9b), we get

$$T_h^{(1)} + T_h^{(2)} = F(x, y; u) + \Upsilon^{(1)} + \Upsilon^{(2)} \ ; \ (x, y) \in \omega_h ; \qquad u|_{G_h} = 0 . \tag{10}$$

Notice that, in above equation we used the fact that function u(x,y) satisfys equation (1). With $\bar{u}$ (x,y) we denote approximate value of function u(x,y), for $(x, y) \in \omega_h$.

By ignoring of sum $\Upsilon^{(1)} + \Upsilon^{(2)}$ in equation (10), we get next difference sheme

$$L_h\overline{u} = T_h^{(1)}\overline{u} + T_h^{(2)}\overline{u} = F(x,y;\overline{u}) \ ; \ (x,y) \in \omega_h \ ; \qquad u|_{G_h} = 0 \, , \tag{11}$$

or diferently written

$$L_h\overline{u}_{i,j} \equiv \frac{\gamma_1}{\Delta c_i^{(1)}}\left[a_{i+1}^{(1)}(\overline{u}_{i+1,j} - \overline{u}_{i,j}) - a_i^{(1)}(\overline{u}_{i,j} - \overline{u}_{i-1,j}) \right] + \frac{\gamma_{21}}{\Delta c_j^{(2)}}\left[a_{j+1}^{(2)}(\overline{u}_{i,j+1} - \overline{u}_{i,j}) - a_j^{(2)}(\overline{u}_{i,j} - \overline{u}_{i,j-1}) \right] =$$

$$= F(x_i, y_j; \overline{u}_{i,j}), \quad (i = 1,2,\ldots, N_1 - 1 \, ; \, j = 1,2,\ldots, N_2 - 1) \ ,$$

$$\overline{u}_{0,j} = \overline{u}_{N_1,j} = 0 \quad (j = 0,1,\ldots,N_2), \qquad \overline{u}_{i,0} = \overline{u}_{i,N_2} = 0 \quad (i = 0,1,\ldots,N_1) \, . \tag{12}$$

Example

Apply this method on solving of Dirichlet's boundary problem

$$0{,}01 \, \Delta \, u(x,y) = x^2 - y^2 \, , \quad \text{on } D = [-1,1] \times [-1,1] \, ,$$

$$u(x,y)|_G = 0 \qquad \textit{(G –bound of area D) .} \tag{13}$$

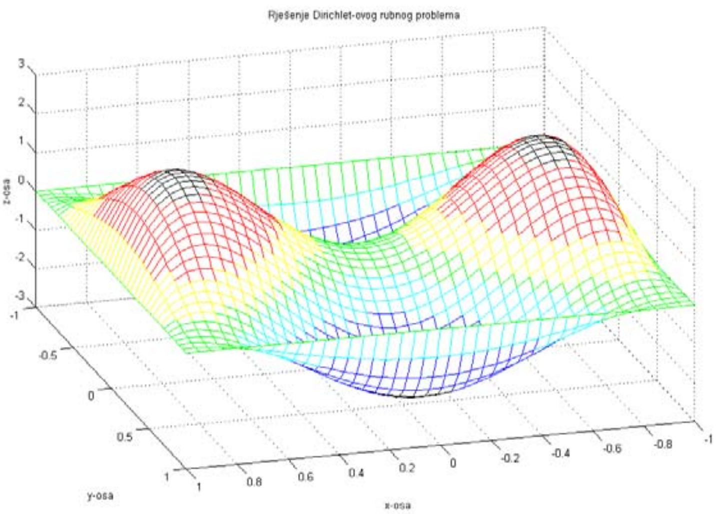

Figure 1. Approximate solution of problem (13)

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