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DIFFERENCE SCHEME FOR SEMILINEAR REACTION-DIFFUSION PROBLEM

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ABSTRACT
We consider the singularly perturbed selfadjoint one-dimensional semilinear reaction-diffusion problem

$$L_\varepsilon y := \varepsilon^2 y''(x) = f(x,y), \quad on \quad (0,1)$$
$$y(0)=0\,; \quad y(1)=0\,,$$

where f(x,y) is a non-linear function. For this problem, using the spline-method with the natural choice of functions, a new difference scheme is given on a non-uniform mesh. The constructed non-linear difference scheme has uniform convergence in points of uneven division segments. A numerical example is given.
Key words. Semilinear reaction-diffusion problem, difference scheme, singular perturbation problem

1. INTRODUCTION

We consider the semilinear singularly perturbed problem:

$$\varepsilon^2 y''(x) = f(x,y), \text{ on } (0,1) \tag{1}$$
$$y(0)=0 \;; \quad y(1)=0\,, \tag{2}$$

where $0<\varepsilon<1$. Assume that the nonlinear function $f(x,y)$ is continuously differentiable, and that it has a strictly positive derivative with respect to y, etc.

$$\frac{\partial f}{\partial y} = f_y \geq m > 0 \quad \text{on} \quad [\;0,1\;]\,x\,R \quad \text{(m=const.)} \tag{3}$$

A solution y of (1) – (2) usually exhibits sharp boundary layers at the endpoints of (0,1), when the parameter ε is near zero. When classical numerical methods are applied to (1)-(2), one does not obtain accurate results on the entire interval (0,1), because we shall use nonstandard discretisation of (1)-(2).

2. CONSTRUCTION OF THE NONLINEAR DIFFERENCE SCHEME

Let us write the differential equation (1) in an equivalent form

$$L_\varepsilon\, y(x) := \varepsilon^2 y''(x) - \gamma\; y(x) = \psi(x,y), \quad \text{on } [0,1], \tag{4}$$

where $\psi(x,y) = f(x,y) - \gamma\, y$, and $\gamma \ge m$ is a chosen constant. Consider an arbitrary grid $0 = x_0 < x_1 < x_2 < ... < x_N = 1$, and consider the following boundary problems

$$L_\varepsilon u_i(x) := 0 \ , \ \text{on} \ (x_i, x_{i+1});\ u_i(x_i) = 1,\ u_i(x_{i+1}) = 0,\ (i = 0,1,...,\ N-1) \qquad (5a)$$

$$L_\varepsilon u_i(x) := 0 \ , \ \text{on} \ (x_i, x_{i+1});\ u_i(x_i) = 0,\ u_i(x_{i+1}) = 1,\ (i = 0,1,...,\ N-1) \qquad (5b)$$

We denote the solutions of problems (5a) and (5b) by $u_i^I(x),\ u_i^{II}(x)\ (i = 0,1,...,\ N-1)$, respectively.

Consider a new boundary problem

$$L_\varepsilon\, y_i(x) = \psi\ (x, y_i)\ ,\ \text{on}\ (x_i, x_{i+1}) \qquad (i = 0,1,...,\ N-1) \qquad (6)$$

$$y_i(x_i) = y\ (x_i) \qquad ; \qquad y_i(x_{i+1}) = y\ (x_{i+1}).$$

Clearly, we have $y_i(x) \equiv y(x)$ on $[0,1]$ $(i = 0,1,...,\ N-1)$. The solution of (6) is given by

$$y_i(x) = C_1 u_i^I(x) + C_2 u_i^{II}(x) + \int_{x_i}^{x_{i+1}} G_i(x,s)\,\psi(s, y(s))ds, \quad (x \in [x_i, x_{i+1}]),$$

where $G_i(x,s)$ is the Green's function associated with the operator L_ε on segment $[x_i, x_{i+1}]$. The function $G_i(x,s)$ in this case has the following form

$$G_i(x,s) = \frac{1}{\varepsilon^2 w_i(s)} \begin{cases} u_i^{II}(x)\, u_i^I(s) \ ; & x_i \le x \le s \le x_{i+1} \\ u_i^I(x)\, u_i^{II}(s) \ ; & x_i \le s \le x \le x_{i+1} \end{cases}, \qquad (7)$$

where $w_i(s) = u_i^{II}(s)\left(u_i^I(x)\right)'_{x=s} - u_i^I(s)\left(u_i^{II}(x)\right)'_{x=s}$.

Clearly, $w_i(s) \ne 0 \quad (s \in [x_i, x_{i+1}])$, because solutions u_i^I and u_i^{II} are linearly independent .

From boundary conditions in (6), it follows that $C_1 = y(x_i) = y_i, C_2 = y(x_{i+1}) = y_{i+1}$ $(i = 0,1,...,\ N-1)$.

Hence, solution y_i of (6) on the segment $[x_i, x_{i+1}]$ has the following form

$$y_i(x) = y_i u_i^I(x) + y_{i+1} u_i^{II}(x) + \int_{x_i}^{x_{i+1}} G\ (x,s)\psi\ (s, y(s))ds\ . \qquad (8)$$

Functions $u_i^I(x)$ and $u_i^{II}(x)$ are known from earlier papers (see e.g. $[2]$), and have forms

$$u_i^I(x) = \frac{\sinh(\beta(x_{i+1} - x))}{\sinh(\beta h_i)}\ ,\ u_i^{II}(x) = \frac{\sinh(\beta(x - x_i))}{\sinh(\beta h_i)} \qquad (x \in [x_i, x_{i+1}]), \qquad (9)$$

$(i = 0,1,...,\ N-1)$, where $\beta = \frac{\sqrt{\gamma}}{\varepsilon}$, $h_i = x_{i+1} - x_i$.

Boundary problem: $L_\varepsilon\, y(x) := \psi\ (x, y)$ on $(0,1)$, $y(0) = y(1) = 0$, (10)

has a unique continuously differentiable solution $y(x)$. Since $y_i(x) \equiv y(x)$ na$[x_i, x_{i+1}]$, ($i = 0,1,...,\ N-1$) we have that $y_i'(x)|_{x=x_i} = y_{i-1}'(x)|_{x=x_i}$, $(i = 1,2,...,\ N-1)$.

Now, differentiating (8), and also by $y_i'(x)|_{x=x_i} = y_{i-1}'(x)|_{x=x_i}$, $(i = 1,2,...,\ N-1)$, we get

$$y_{i-1}\left(u_{i-1}^I(x)\right)'_{x=x_i} + y_i\left[\ \left(u_{i-1}^{II}(x)\right)'_{x=x_i} - \left(u_i^I(x)\right)'_{x=x_i}\ \right] + y_{i+1}\left[-\left(u_i^{II}(x)\right)'_{x=x_i}\ \right] =$$

$$= \frac{d}{dx}\left[\ \int_{x_i}^{x_{i+1}} G_i(x,s)\psi\ (s, y(s))ds - \int_{x_{i-1}}^{x_i} G_{i-1}(x,s)\psi\ (s, y(s))ds\ \right]_{x=x_i}. \qquad (11)$$

where $y_k = y(x_k) \quad (k = i-1, i, i+1)$.

We define $a_i = -\left(u_{i-1}^I(x)\right)'_{x=x_i}$; $d_i = \left(u_{i-1}^{II}(x)\right)'_{x=x_i} - \left(u_i^I(x)\right)_{x=x_i}$; $b_i = -\left(u_i^I(x)\right)'_{x=x_i}$.

From (9) it follows that : $a_i = \dfrac{\beta}{\sinh(\beta h_{i-1})}$; $b_i = \dfrac{\beta}{\sinh(\beta h_i)}$; $d_i = \dfrac{\beta}{\tanh(\beta h_{i-1})} + \dfrac{\beta}{\tanh(\beta h_i)}$.

Hence, now (11) has the following form:

$$-a_i y_{i-1} + d_i y_i - b_i y_{i+1} = = \int_{x_i}^{x_{i+1}} \frac{d}{dx}\left(G_i(x,s)\right)_{x=x_i} \psi\left(s, y(s)\right) ds - \int_{x_{i-1}}^{x_i} \frac{d}{dx}\left(G_{i-1}(x,s)\right)_{x=x_i} \psi\left(s, y(s)\right) ds .$$

After differentiating, the right hand side in the above equation becomes

$$a_i y_{i-1} - d_i y_i + b_i y_{i+1} = \frac{1}{\varepsilon^2}\left[\int_{x_{i-1}}^{x_i} u_{i-1}^{II}(s)\psi\left(s, y(s)\right) ds + \int_{x_i}^{x_{i+1}} u_i^I(s)\psi\left(s, y(s)\right) ds\right],$$

$$y_i = 0 \quad ; \quad y_N = 0 \quad \text{za} \quad \left(i = 0,1,\ldots, N-1\right) . \qquad (12)$$

Clearly, we cannot generally explicitly compute the integrals in (12). We approximate the function $\psi\left(x, y(x)\right)$, on the segment $\left[x_{i-1}, x_i\right]$, by

$$\overline{\psi}_{i-1} = \overline{\psi}\left(x, y(x)\right) = \psi\left(\frac{x_{i-1} + x_i}{2}, \frac{\overline{y}_{i-1} + \overline{y}_i}{2}\right) \quad \left(na\ \left[x_{i-1}, x_i\right]\right) \quad (i = 1,2,\ldots,N) ,$$

where $\overline{y}_i$ $\left(i = 1,2,\ldots, N-1\right)$ are approximation values of the solution $y(x)$ of the problem (1) – (3) in points x_i $\left(i = 1,2,\ldots, N-1\right)$. Finally from (12) we get the difference scheme

$$a_i \overline{y}_{i-1} - d_i \overline{y}_i + b_i \overline{y}_{i+1}^I = \frac{1}{\varepsilon^2}\left[\overline{\psi}_{i-1} \int_{x_{i-1}}^{x_i} u_{i-1}^{II}\, ds + \overline{\psi}_i \int_{x_i}^{x_{i+1}} u_i^I\, ds\right] \left(i = 1,2,\ldots, N-1\right).$$

From (9), we have

$$\int_{x_{i-1}}^{x_i} u_{i-1}^{II}(s) ds = \frac{1}{\beta}\cdot\frac{\cosh(\beta h_{i-1})}{\sinh(\beta h_{i-1})} - \frac{1}{\beta}\cdot\frac{1}{\sinh(\beta h_{i-1})}, \quad \int_{x_i}^{x_{i+1}} u_i^I(s) ds = \frac{1}{\beta}\cdot\frac{\cosh(\beta h_i)}{\sinh(\beta h_i)} - \frac{1}{\beta}\cdot\frac{1}{\sinh(\beta h_i)}.$$

Hence, our difference scheme has the following form:

$$a_i \overline{y}_{i-1} - d_i \overline{y}_i + b_i \overline{y}_{i+1} = \frac{1}{\gamma}\overline{\psi}_{i-1}(c_i - a_i) + \frac{1}{\gamma}\overline{\psi}_i(c_{i+1} - a_{i+1}), \text{ where is } c_i = \frac{\beta}{tgh(\beta h_{i-1})} .$$

We define $c_i - a_i = \Delta c_i$ i $c_{i+1} - a_{i+1} = \Delta c_{i+1}$. We can write the last difference scheme in the form

$$a_i \overline{y}_{i-1} - (c_i + c_{i+1})\overline{y}_i + a_{i+1}\overline{y}_{i+1} = \frac{1}{\gamma}\overline{\psi}_{i-1}\Delta c_i + \frac{1}{\gamma}\overline{\psi}_i \Delta c_{i+1} \qquad (14)$$

where $b_i = a_{i+1}$ and $d_i = c_i + c_{i+1}$. Since $\psi(x,y) = f(x,y) - \gamma y$, from (14) we have:

$$a_i \overline{y}_{i-1} - (c_i \overline{y}_i + c_{i+1}\overline{y}_i) + a_{i+1}\overline{y}_{i+1} + \frac{\Delta c_i}{2}\left(\overline{y}_{i-1} + \overline{y}_i\right) + \frac{\Delta c_{i+1}}{2}\left(\overline{y}_i + \overline{y}_{i+1}\right) = \frac{\Delta c_i}{\gamma}\overline{f}_{i-1} + \frac{\Delta c_{i+1}}{\gamma}\overline{f}_i$$

where $\overline{f}_{i-1} = f\left(\dfrac{x_{i-1} + x_i}{2}, \dfrac{\overline{y}_{i-1} + \overline{y}_i}{2}\right)$. After some computation, we get:

$$\left(\frac{a_i + c_i}{2}\right)\overline{y}_{i-1} - \left(\frac{c_i + a_i}{2} + \frac{c_{i+1} + a_{i+1}}{2}\right)\overline{y}_i + \left(\frac{c_{i+1} + a_{i+1}}{2}\right)\overline{y}_{i+1} = \frac{\Delta c_i}{\gamma}\overline{f}_{i-1} + \frac{\Delta c_{i+1}}{\gamma}\overline{f}_i$$

If we define $\dfrac{a_i + c_i}{2} = r_i$, then the difference scheme (14) gets the simpler form:

$$r_i \overline{y}_{i-1} - \left(r_i + r_{i+1}\right)\overline{y}_i + r_{i+1}\overline{y}_{i+1} = \frac{\Delta c_i}{\gamma}\overline{f}_{i-1} + \frac{\Delta c_{i+1}}{\gamma}\overline{f}_i . \qquad (15)$$

3. NUMERICAL RESULTS

Consider the following boundary problem

$$\varepsilon^2 y'' = y^3 + y - 10, \qquad \text{for} \quad x \in (0,1)$$
$$y(0) = y(1) = 0.$$

We use difference scheme (15) to compute the approximate solution.

Table 1. Error E_n , and convergence rates Ord for approximate solution

	$\varepsilon = 2^{-4}$	$\varepsilon = 2^{-7}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-16}$	$\varepsilon = 2^{-22}$	$\varepsilon = 2^{-28}$	$\varepsilon = 2^{-30}$	
$N = 64$	1.54e-3	1.73e-3	1.76e-3	1.76e-3	1.76e-3	1.76e-3	1.76e-3	*En*
	2.05	2.01	2.01	2.01	2.01	2.01	2.01	*Ord*
$N = 128$	3.83e-4	4.30e-4	4.36e-4	4.36e-4	4.37e-4	4.37e-4	4.37e-4	*En*
	2.01	2.00	2.00	2.00	2.00	2.00	2.00	*Ord*
$N = 256$	9.58e-5	1.07e-4	1.09e-4	1.09e-4	1.09e-4	1.09e-4	1.09e-4	*En*
	2.00	2.00	2.00	2.00	2.00	2.00	2.00	*Ord*
$N = 512$	2.39e-5	2.68e-5	2.72e-5	2.72e-5	2.72e-5	2.72e-5	2.72e-5	*En*

$E_n = \max_{1 \le i \le n} \left| (y^n)_i - \left(y^{2n}\right)_i \right|$, where is $(y^{(n)})_i = \overline{y}(x_i)$, approximate values of the unknown function y, in *i*-th points of mesh, n is the number of points in the mesh. The convergence rate (Ord) is defined by $Ord = \dfrac{\ln(E_n) - \ln(E_{2n})}{\ln 2}$.

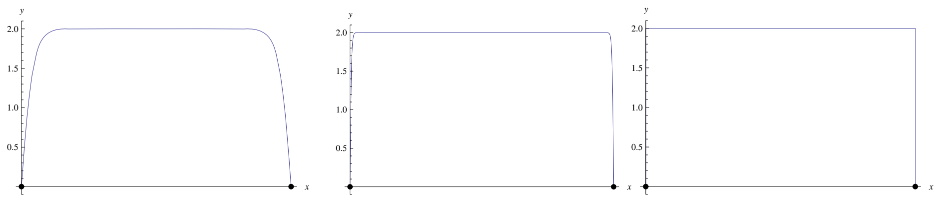

Figure 1. Graphics approximates solutions for values $\varepsilon = 2^{-4}$, $\varepsilon = 2^{-7}$ and $\varepsilon = 2^{-30}$

4. REFERENCES

[1] N.Okičić, E.Duvnjaković, *Difference scheme for nonlinear singular perturbation problem,* Godišnjak, Faculty of Science Tuzla, pp. 53-60, 2004.

[2] E.Duvnjaković, *A Class of Difference schemes for singular perturbation problem*, Proceedings of the 7 International Conference on Operational research, Croatian OR Society, pp. 197-208, 1999.

[3] H.G.Roos, M.Stynes, L.Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations,*Springer-Verlag Berlin Heidelberg, 2008.g.