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THE RATIONAL SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS IN THE MODELING COMPETITIVE POPULATIONS

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ABSTRACT
In a modelling setting, the rational system of nonnlinear difference equations

$$x_{n+1} = \frac{x_n}{a + y_n^2}, \quad y_{n+1} = \frac{y_n}{b + x_n^2}, \quad n = 0,1,\ldots$$

represents the rule by which two discrete, competitive populations reproduce from one generation to the next. The phase variables x_n and y_n denote population sizes during the n-th generation and sequence or orbit $\{(x_n, y_n): \quad n = 0,1,\ldots\}$ describes how the populations evolve over time. Competitive between the populations is reflected by the fact the transition function for each population is a decreasing function of the other population size.
In this paper we will investigate the rate of convergence of a solution that convergence to the equilibrium $(0,0)$ of a rational system of difference equations where the parameters a and b are positive numbers, and conditions x_0 and y_0 are arbitrary nonnegative numbers.
Key words: difference equations, global stability, rate of convergence.[2]

1 INTRODUCTION

The system of a difference equations

$$x_{n+1} = \frac{x_n}{a + y_n^2}, \quad y_{n+1} = \frac{y_n}{b + x_n^2}, \quad n = 0,1,\ldots, \tag{1}$$

where the parameters *a* and *b* are positive numbers, and initial conditions x_0 and y_0 are arbitrary nonnegative numbers, has been investigated in [1]. The equilibrium points $(\bar{x}, \bar{y})$ of a system (1) satisfy the system of equations

$$\bar{x} = \frac{\bar{x}}{a + \bar{y}^2}, \quad \bar{y} = \frac{\bar{y}}{b + \bar{x}^2}, \quad n = 0,1,\ldots. \tag{2}$$

The equilibrium of system (1) are $E_0 = (0,0)$ for positive values parameters *a* and *b*, and $E_{a,b} = \left(\sqrt{1-b}, \sqrt{1-a}\right)$ for $a \le 1$ and $b \le 1$, where at least one inequality is strict. Our linearized stabi-

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ty analysis indicates several (six) cases with different asymptotic behavior depending on the values of parameters *a* and *b*.
The following global asymptotic stability result has been obtained in [1].

Theroem 1.1 *Assume that* $a>1$ *and* $b>1$. *Then the equilibrium point* $(0,0)$ *is a globally asymptotically stable, i.e. every solution* $\{(x_n, y_n)\}$ *of system (1) satisfies*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0 .$$

The global stable manifold $W^s((0,0)) = \{(x,y) : x \ge 0, y \ge 0\}$.

Our goal is a to investigate the rate of convergence of solution of a system (1) that converges to the equilibrium $E_0 = (0,0)$ in the regions parameters described in Theorem 1.1. The rate of convergence of solutions that convergence to an equilibrium has been obtanied for some two-dimensional system in [5] and [6]. The following results gives the rate of convergence of solutions of a systema difference equations

$$\mathbf{x}_{n+1} = [A + B(n)]\mathbf{x}_n , \qquad (3)$$

where $\mathbf{x}_n$ is a *k*-dimensional vectors, $A \in C^{k\times k}$ is a constans matrix, and $B : \mathbf{Z}^+ \to C^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to \infty , \qquad (4)$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm.

Theorem 1.2 ([8]) *Assume that condition (4) hold. If* $\mathbf{x}_n$ *is a solution of system (3), then either* $\mathbf{x}_n = \mathbf{0}$ *for all large n or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|} \qquad (5)$$

exist and is equal to the moduls of one the eigenvalues of matrix A.

Theorem 1.3 ([8]) *Assume that condition (4) hold. If* $\mathbf{x}_n$ *is a solution of system (3), then either* $\mathbf{x}_n = \mathbf{0}$ *for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|} \qquad (6)$$

exist and is equal to the moduls of one the eigenvalues of matrix A.

2 RATE OF CONVERGENCE

In this section we will determinate the rate of convergence of a solution that converges to the equilibrium $E_0 = (0,0)$, in case describe in Theorem 1.1. But, we will prove this generally theorem.

Theorem 2.1 *Assume that a solution* $\{(x_n, y_n)\}$ *of a system (1) converges to the equilibrium* $E = (\bar{x}, \bar{y})$ *and E is globally asymptotically stable. The error vector*

$$\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \end{pmatrix}$$

of every solution $\mathbf{x}_n \ne \mathbf{0}$ *of (1) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \quad \textit{for some } i = 1,2 , \qquad (7)$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \quad \textit{for some } i = 1,2 \qquad (9)$$

where $\left|\lambda_i\left(J_T\left(E\right)\right)\right|$ *is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium* $J_T\left(E\right)$.

Prof. First we will find a system satisfied by the error terms. The error terms are given

$$x_{n+1}-\overline{x}=\frac{x_n}{a+y_n^2}-\frac{\overline{x}}{a+\overline{y}^2}=\frac{x_n\left(a+\overline{y}^2\right)-\overline{x}\left(a+y_n^2\right)}{\left(a+y_n^2\right)\left(a+\overline{y}^2\right)}=\frac{\left(x_n-\overline{x}\right)\left(a+\overline{y}^2\right)-\overline{x}\left(y_n^2-\overline{y}^2\right)}{\left(a+y_n^2\right)\left(a+\overline{y}^2\right)}$$

$$=\frac{1}{a+y_n^2}\left(x_n-\overline{x}\right)-\frac{\left(y_n+\overline{y}\right)}{\left(a+y_n^2\right)}\frac{\overline{x}}{\left(a+\overline{y}^2\right)}\left(y_n-\overline{y}\right)=\frac{1}{a+y_n^2}\left(x_n-\overline{x}\right)+\frac{-\overline{x}\left(y_n+\overline{y}\right)}{a+y_n^2}\left(y_n-\overline{y}\right),$$

and

$$y_{n+1}-\overline{y}=\frac{y_n}{b+x_n^2}-\frac{\overline{y}}{b+\overline{x}^2}=\frac{y_n\left(b+\overline{x}^2\right)-\overline{y}\left(b+x_n^2\right)}{\left(b+x_n^2\right)\left(b+\overline{x}^2\right)}=\frac{\left(y_n-\overline{y}\right)\left(b+\overline{x}^2\right)-\overline{y}\left(x_n^2-\overline{x}^2\right)}{\left(a+y_n^2\right)\left(a+\overline{y}^2\right)}$$

$$=\frac{1}{b+x_n^2}\left(y_n-\overline{y}\right)-\frac{\left(x_n+\overline{x}\right)}{\left(b+x_n^2\right)}\frac{\overline{y}}{\left(b+\overline{x}^2\right)}\left(x_n-\overline{x}\right)=\frac{1}{b+x_n^2}\left(y_n-\overline{y}\right)+\frac{-\overline{y}\left(x_n+\overline{x}\right)}{b+x_n^2}\left(x_n-\overline{x}\right).$$

That is

$$\left.\begin{aligned}x_{n+1}-\overline{x}&=\frac{1}{a+y_n^2}\left(x_n-\overline{x}\right)+\frac{-\overline{x}\left(y_n+\overline{y}\right)}{a+y_n^2}\left(y_n-\overline{y}\right),\\ y_{n+1}-\overline{y}&=\frac{1}{b+x_n^2}\left(y_n-\overline{y}\right)+\frac{-\overline{y}\left(x_n+\overline{x}\right)}{b+x_n^2}\left(x_n-\overline{x}\right).\end{aligned}\right\} \tag{9}$$

Set

$$e_n^1=x_n-\overline{x} \text{ and } e_n^2=y_n-\overline{y}.$$

Then system (9) can be represented as

$$e_{n+1}^1=a_ne_n^1+b_ne_n^2,$$
$$e_{n+1}^2=d_ne_n^1+c_ne_n^2,$$

where

$$a_n=\frac{1}{a+y_n^2},\ b_n=\frac{-\overline{x}\left(y_n+\overline{y}\right)}{a+y_n^2},\ c_n=\frac{1}{b+x_n^2},\ d_n=\frac{-\overline{y}\left(x_n+\overline{x}\right)}{b+x_n^2}.$$

Taking the limits of a_n, b_n, c_n and d_n, we obtain

$$\lim_{n\to\infty}a_n=\frac{1}{a+\overline{y}^2},\ \lim_{n\to\infty}b_n=-\frac{2\overline{x}\,\overline{y}}{a+\overline{y}^2},$$
$$\lim_{n\to\infty}c_n=\frac{1}{b+\overline{x}^2},\ \lim_{n\to\infty}d_n=-\frac{2\overline{x}\,\overline{y}}{b+\overline{x}^2},$$

that is

$$a_n=\frac{1}{a+\overline{y}^2}+\alpha_n,\ b_n=-\frac{2\overline{x}\,\overline{y}}{a+\overline{y}^2}+\beta_n,$$
$$c_n=\frac{1}{b+\overline{x}^2}+\gamma_n,\ d_n=-\frac{2\overline{x}\,\overline{y}}{b+\overline{x}^2}+\delta_n,$$

where

$$\alpha_n\to 0, \beta_n\to 0, \gamma_n\to 0 \text{ and } \delta_n\to 0 \text{ when } n\to\infty.$$

Now we have system of the form (3):

$$\mathbf{e}_{n+1}=\left[A+B(n)\right]\mathbf{e}_n.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e_{n+1}^1 \\ e_{n+1}^2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{a+\overline{y}^2} & -\dfrac{2\overline{x}\,\overline{y}}{a+\overline{y}^2} \\ -\dfrac{2\overline{x}\,\overline{y}}{b+\overline{x}^2} & \dfrac{1}{b+\overline{x}^2} \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix}.$$

The system is exactly linearized system of (1) evaluted at the equilibrium $E=(\overline{x},\overline{y})$. Then Theorems 1.2 and 1.3 imply the result. ■

If we get $E=(\overline{x},\overline{y})=(0,0)$, then we obtain the following result.

Corollary 2.1 *Assume that* $a>1$ *and* $b>1$. *Then the equilibrium point* $E=(\overline{x},\overline{y})=(0,0)$ *is a globally asymptotically stable. The error vector of every solution* $\mathbf{x}_n \neq \mathbf{0}$ *of (1) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = \lim_{n\to\infty} \sqrt[n]{x_n^2+y_n^2} = \left|\lambda_i\left(J_T(E)\right)\right| \text{ for some } i=1,2,$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = \lim_{n\to\infty} \sqrt{x_{n+1}^2+y_{n+1}^2} = \left|\lambda_i\left(J_T(E)\right)\right| \text{ for some } i=1,2,$$

where $\left|\lambda_i\left(J_T(E)\right)\right|$ *is equal to the moduls of one the eigenvalues of the Jacobian matrix evaluted at the equilibrium* $J_T(E)$ *i.e.* $\lambda_i \in \left\{\frac{1}{a},\frac{1}{b}\right\}$.

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